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Un teorema per la sottobase $\{B\}$

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For the base $\{B\}$ we have the following result:

THEOREM 4.1. - For any X in $\mathcal{L}(\{B\})$ we may construct w_o in $\{B\}^+$ corresponding to X, such that: $\forall$ w in corresponding to X, $SL(w_o) \leq SL(w)$.

PROOF. We shall prove that the combinator w_o corresponding to X, is obtained by the elimination of the rightmost parenthesis that can be eliminated at every expansion step by using one B. Let us consider the class S of all the strategies that reach wx $x_2 \ldots x_n$ from X.

We may notice first that in an expansion step of a given strategy $s \in S$ the elimination of a parenthesis introduces as "side-effect" some other parentheses. For instance, in the elimination of the parenthesis (α) of ξ, we introduce the parentheses (β), (γ) and (δ):

$$\xi = BB\, x_1 \underset{(\alpha)}{(x_2x_3)(x_4x_5)} \leq B(\underset{(\delta)}{(}\underset{(\gamma)}{(}\underset{(\beta)}{(}BB)x_1)(x_2x_3))x_4x_5$$

In the future we shall call "extra parentheses" the introduced parentheses surrounding at least one variable (e.g.: (δ) and (γ) are "extra parentheses").

We notice also that, in order to obtain w from a given X, we must remove from X all its parentheses and all extra parentheses introduced at each expansion step. Therefore the strategy by which we may obtain w_o is the one for which the minimum number of extra parentheses is introduced.

Let us now state the following assertation A:

$\forall s \in S$ the strategy $s_o \in S$ that removes the rightmost parenthesis in the achieved formula at each expansion step reaches $wx_1 \ldots x_n$ from X in a not greater number of steps than s.

We shall prove the truth of the assertion A in two steps:

(i) first we prove by structural induction that the assertation A holds for every combination $\tilde{X}$ in $\mathcal{L}(\{B\})$ of the form

$$x_1 \chi_1 \chi_2 \ldots \chi_m$$

where every χ_i is a combination of variables;

(ii) then we prove the assertion for every combination X in $L(\{B\})$.

(i). Let us consider first the combination $\hat{X} = x_1 \underset{(1)}{(x_2x_3)}\underset{(2)}{(x_4x_5)}\ldots\underset{(n)}{(x_{2n}x_{2n+1})}$

Given a strategy s, to obtain a w corresponding to X, we may associate to s a n-tuple $\underline{s} = \langle j_1, j_2, \ldots, j_n\rangle$ where $j_i, 1 \leq j_i \leq n$, is the i-th parenthesis of the set of the initial ones, removed in the strategy s.

If we point out the decreasing subsequences of $\underline{s}$, we may write:

$$\underline{s} = j_1^{(1)}, j_2^{(1)}, \ldots, j_{l_1}^{(1)}; j_1^{(2)}, j_2^{(2)}, \ldots, j_{l_2}^{(2)}; \ldots; j_1^{(k)}, j_2^{(k)}, \ldots, j_{lk}^{(k)}$$

where: a) $\forall i, 1 \leq i \leq k \qquad 1 \leq l_i \leq n$

b) $\forall i, 1 \leq i \leq k-1 \qquad j_{l_i}^{(i)} < j_1^{(i+1)}$

c) $\forall i, 1 \leq i \leq k \qquad j_n^{(i)} > j_m^{(i)}$ where $1 \leq n < m \leq l_i$.

The extra parentheses introduced in the elimination of the parentheses $j_1^{(1)}, j_1^{(2)}, \ldots, j_1^{(k)}$ are at least respectively $j_1^{(1)}-1$, $j_1^{(2)}-j_{l_1}^{(1)}+2, \ldots, j_1^{(k)}-$ $+j_{l_{k-1}}^{(k-1)}+2$.

Moreover in the elimination of the parentheses of each subsequence $\underline{s}^{(p)} = \langle j_2^{(p)}, \ldots, j_{l_p}^{(p)}\rangle$, $1 \leq p \leq k$, of $\underline{s}$ we have to introduce at least $l_p - 1$ new extra parentheses.

Therefore the optimal strategy s_o for $\hat{X}$ is the one for which the following expression e is minimized:

$$e = j_1^{(1)} - 1 + \sum_{2}^{k}{}_i (j_1^{(i)} - j_{l_{i-1}}^{(i-1)} + 2) + \sum_{1}^{k}{}_i (l_i - 1)$$

It is easy to see that the n-tuple $\underline{s}_o = \langle n, n-1, \ldots, 1\rangle$ minimizes e and the strategy s_o, that generates $\underline{s}_o$ by construction, satisfies the assertion A.

We give now the structural induction argument:
whatever is $\tilde{X}$ one may obtain $\tilde{X}$ from $\hat{X}$ by iterative use of the following structural transformation: one variable x_i, where $x_i \neq x_1$, is substituted by the application $(x_j\ x_{j+1})$ and redenomination of variables is made in order to obtain an element of $L(\{B\})$.

The use of the above transformation preserves the truth of the assertion A.

In fact we have two cases:

1) if the replaced x_i is a left-applied object, then in the elimination of the parenthesis surrounding x_j and x_{j+1}, we introduce in the strategy s_o one more extra parenthesis and this is obviously the minimum number possible;

2) if the replaced x_i is a right-applied object, then in the elimination of the parenthesis surrounding x_j and x_{j+1}, we introduce in the strategy s_o two more extra parentheses (that we call (1) and (2)):

$$\underset{(\alpha)}{\xi_\delta (}x_{i-1}\underset{(\beta)}{(}x_j x_{j+1}))\ldots \leq B\,\xi_\delta\, x_{i-}\underset{(\beta)}{(}x_j x_{j+1})\ldots \leq B\underset{(1)(2)}{((}B\xi_\delta)x_{i-1})x_j x_{j+1}\ldots$$

Nevertheless, one parenthesis must be introduced by every strategy.

For the second one a generic strategy in which it is not necessary to introduce at this step a second parenthesis is the following: after the elimination of (α) and before the elimination of (β) we remove at least one parenthesis, say (γ), on the righthand of (α). But in this case when we remove the parenthesis (γ), we have to introduce one more extra parenthesis than in the case of strategy s_o, because the parenthesis (α) does not exist any more.

(ii). To complete the proof of the theorem we have to show that if the assertion A is true for every combination $\tilde{X}$ it is also true for every $X \in L(\{B\})$, that is for every

$$X = x_1 x_{i_1}\ldots x_{j_1} \chi_1 x_{i_2}\ldots x_{j_2} \chi_2 x_{i_3}\ldots x_{j_3} \chi_3 \ldots \chi_{m-1} x_{i_m}\ldots x_{j_m} \chi_m,$$

where χ_i's are combinations of variables.

In the expansion procedure from X to $wx_1\ldots x_n$ we reach the following intermediate formulas:

(0) X

(1) $\xi_{\delta 1} x_{i_m}\ldots x_n$

(2) $\xi_{\delta 2} x_{i_{m-1}}\ldots x_n$

$\vdots$

(m-1) $\xi_{m-1} x_{i_2} \dots x_n$

(m) $w\, x_1 \dots x_n$

For the strategy s_o the only effect of the variables between χ_{m-k-1} and χ_{m-k} (where $0 \leq k \leq m-1$ and $\chi_o = x_1$) is to increase by $j_{m-k} - i_{m-k} + 1$ steps the length of the strategy to reach the (k+1)-th formula from the k-th one.

It is easy to see that every strategy s must reach the formulas (1), (2),...,(m) and it must necessarily increase its length at least as the strategy s_o.

Q.E.D.

As a consequence of theorem 4.1, we may notice that, if $wx \dots x_n \geq X$ and $w'x_1 \dots x_n \geq X'$, where $X, X' \in L(\{B\})$ and $w, w' \in \{B\}^+$:

(i) if X' has a lower number of parentheses to be eliminated(*) than X, then $SL(w') < SL(w)$;

(ii) if X' is obtained from X by moving on the left one couple of parentheses of X to be eliminated, then $SL(w') < SL(w)$.

We can also establish the following:

Theorem 4.2. - For any X in $L(\{B\})$ such that $SL(X) = n$(**) we have that if $w \in \{B\}^+$ corresponds to X, then $SL(w)=O(n)$.

Proof. The structure of X such that $X \in L(\{B\})$ and $SL(X)=n$, in which there is the minimum number (1) of parentheses to be eliminated, is of the form:

$$\bar{X}_n = x_1 \dots x_{n-2}(x_{n-1} x_n)$$

(*) We suppose all parentheses to be eliminated are explicited.

(**) The definitions of structural complexity obviously can be extended to pure combinations.

On the other hand, the structure of X such that $X \in \mathcal{L}(\{B\})$ and $SL(X)=n$, in which there is the maximum number of parentheses to be eliminated and these are in the rightmost position, is of the form:

$$\overline{\overline{X}}_n = x_1(x_2(\ldots(x_{n-1}x_n)\ldots))$$

If can be easily verified that: if $\overline{w}_n x_1 \ldots x_n \geq \overline{X}_n$, then $\overline{w}_{n+1} = B\overline{w}_n$;
if $\overline{\overline{w}}_n x_1 \ldots x_n \geq \overline{\overline{X}}_n$, then $\overline{\overline{w}}_{n+1} = B(B\overline{\overline{w}}_n)B$;
$\overline{w}_3 = \overline{\overline{w}}_3 = B$.

Q.E.D.