The Solution of Böhm's S-Problem

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In this note we give an example of a term of Combinatory Logic (CL-term) in which only the combinator S occurs and which does not have a (weak) normal form. This answers a question of C. Böhm, put at the Symposium on λ-Calculus and Computer Science Theory, held at Rome, march 1975.

<u>1. Definition</u>: For arbitrary CL-terms A,B A^nB is defined inductively by

$$A^0B = B \quad , \quad A^{n+1}B = A(A^nB) \quad .$$

Notations are as in Curry-Feys [2] ; [,], { , } are sometimes used instead of (,) to make things easier to read.

<u>2. Lemma</u>: Let $C = S(SS)$ and B be an arbitrary term. Then for all natural numbers n,m

$T(n,m) =_{Df.} (C^n(BC))(C^m(BC))$

reduces to a term T' which has $BC(C^m(BC))$ as a subterm.

<u>Proof</u>: By induction on n .

$T(0,m) = BC(C^m(BC))$. Assume the hypothesis is true for $T(n,m)$. Then

$$\begin{aligned} T(n+1,m) &= (C(C^n(BC)))(C^m(BC)) \\ &= S(SS)(C^n(BC))(C^m(BC)) \\ &\geq SS(C^m(BC))[(C^n(BC))(C^m(BC))] \\ &= SS(C^m(BC))\ T(n,m) \quad . \end{aligned}$$

But $T(n,m)$ has the desired property.

3. Corollary: If in addition to the assumptions in 2. $B = C = S(SS)$ then for all m $T(0,m)$ reduces to a term which has $T(0,m+1)$ as a subterm.

Proof:
$$\begin{aligned} T(0,m) &= BC(C^m(BC)) \\ &= S(SS)C(C^m(BC)) \\ &\geq SS(C^m(BC))[C^{m+1}(BC)] \\ &\geq S(C^{m+1}(BC))[(C^m(BC))(C^{m+1}(BC))] \\ &= S(C^{m+1}(BC))\ T(m,m+1) \ . \end{aligned}$$
Now apply the lemma to $T(m,m+1)$.

4. Theorem: None of the terms $T(n,m)$ as in 3. has a normal form.

Proof: For each $T(n,m)$ we can construct from the above an infinite reduction sequence
$T(n,m) = T_1 \geq T_2 \geq \ldots \geq T_k \geq \ldots .$.
As we have only S-redeces the redex in T_k which is contracted to obtain T_{k+1} will have a residual in any other term T with $T_k \geq T$. Therefore $T(n,m)$ cannot have a normal form.

5. Corollary: Let $A = B = C = S(SS)$. Then $SABC$ has no normal form.

Proof:
$$\begin{aligned} SABC &\geq AC(BC) \\ &= S(SS)C(BC) \\ &\geq SS(BC)(C(BC)) \\ &\geq S(C(BC))\{BC(C(BC))\} \\ &= S(C(BC))\ T(0,1) \end{aligned}$$
and $T(0,1)$ has no normal form.

Remarks: 1. Naturally it was our original idea to prove corollary 5. We conjecture that if $S^1 = S$ and $S^{n+1} = S(S^n)$ for no $n \geq 3$ $SS^nS^nS^n$ has a normal form.

2. After we had finished the above proofs Henk Barendregt informed us that he too had solved Böhm's problem. His example is as follows: Let $\not S = SSS$, then $f = \not S \not S \not S$ has no normal form. For a motivation for his notation see [1].

References:

[1] Barendregt, H. ; Letter to the authors from april 14th, 75

[2] Curry, H.B. ; Feys, R.; Combinatory Logic I
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Fachbereich 4