ON THE EXPRESSIVE POTENTIALITIES OF DEDUCTIVE SYSTEMS OF λ-CONVERSION AND COMBINATORY LOGIC

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Vestnik Moskovskogo Universiteta. Matematika, Vol. 29, No. 6, pp. 19-26, 1974

UDC 517.12

We shall investigate the combinatorially complete systems, proposed in [1-3], from the point of view of their expressive potentialities. We shall retain the definitions, abbreviations, and notation employed in [1-3].

1. THE VARIANT OF A SYSTEM OF λ-CONVERSION WITH THE OPERATOR OF FORMAL IMPLICATION

As the initial (undefined objects (obs) we take the operator of formal implication Ξ and variables (we denote variables by the letters x, y, z, t, possibly with subscripts). We generate new obs from those already constructed by applying the operator of application and the operator of abstraction λ: if a and b are obs, and x a variable, then (ab) and (λxa) will be regarded as obs. We denote obs by the letters a,...,h of the Latin alphabet (possibly with subscripts). The abstraction operator λ is a unique operator, connecting the variables in obs. The expressions "$x_1, \dots, x_n \overline{\in} a_1, \dots, a_m$" means that the variables $x_1,\dots,x_n$ have no free embeddings in the obs $a_1,\dots,a_m$. The result of simultaneously substituting $a_1,\dots,a_n$ in the ob b instead of the corresponding free embeddings of the different variables $x_1,\dots,x_n$ is denoted by $[a_1, \dots, a_n/x_1, \dots, x_n]b$.

Words of the form $a \to b$ and $\Gamma \Rightarrow \Delta$ will be regarded as sequents, the first being a λ-sequent and the second a deductive sequent, while the letters $\Gamma, \Delta, \theta, \Lambda$ will be used to denote collections (possibly empty) of obs. For brevity, some brackets in obs will be omitted; they are restored according to the principle: "brackets to the left."

Postulates of λ-conversion. α: $\lambda xa \to \overline{\lambda} y[y/x]a\ (y \in a)$; β: $(\lambda xa)\,b \to [b/x]\,a$;

$$\mu: \frac{a \to b}{ca \to cb};\quad \nu: \frac{a \to b}{ac \to bc};\quad \xi: \frac{a \to b}{\lambda xa \to \lambda xb};\quad \tau: \frac{a \to b;\ b \to c}{a \to c};\quad \sigma: \frac{a \to b}{b \to a}.$$

Here are formulated the schemes of axioms α and β, and the rules of inference $\mu, \nu, \xi, \tau, \sigma$ of the calculus of λ-conversion; for details of λ-conversion see [4,5] and the references quoted in [1,2].

Postulates of extension.

$$1.\ \frac{a \to b}{a \Rightarrow b};\quad 2.\ \frac{\Gamma \Rightarrow \theta}{\Gamma \Rightarrow \theta, a};\quad 3.\ \frac{\Gamma \Rightarrow \theta}{a, \Gamma \Rightarrow \theta};\quad 4.\ \frac{\Gamma \Rightarrow \theta, a, a}{\Gamma \Rightarrow \theta, a};\quad 5.\ \frac{a, a, \Gamma \Rightarrow \theta}{a, \Gamma \Rightarrow \theta};$$

$$6.\ \frac{\Gamma \Rightarrow \Lambda, a, b, \theta}{\Gamma \Rightarrow \Lambda, b, a, \theta};\quad 7.\ \frac{\Delta, a, b, \Gamma \Rightarrow \theta}{\Delta, b, a, \Gamma \Rightarrow \theta};\quad 8.\ \frac{\Gamma \Rightarrow \theta, a;\ a \to b}{\Gamma \Rightarrow \theta, b};$$

$$9.\ \frac{a, \Gamma \Rightarrow \theta;\ b \to a}{b, \Gamma \Rightarrow \theta};\quad 10.\ \frac{ax, \Gamma \Rightarrow \theta, bx;\ x \overline{\in} a, b, \Gamma, \theta}{\Gamma \Rightarrow \theta, \Xi ab};\quad 11.\ \frac{\Delta \Rightarrow \Lambda, ac;\ bc, \Gamma \Rightarrow \theta}{\Xi ab, \Delta, \Gamma \Rightarrow \Lambda, \theta}.$$

We shall call the system described a λΞ-system; if only single-succeedent sequents participate in the postulates of extension, we shall refer to a λΞ*-system (cf. the systems in [2,3]). Rules 8 and 9 are the rules of λ-cut (cf. the combinatorial rules in [1,2]).

Definitions. 1. We regard an ob of the form (λxa)b as a redex, and an ob of the form [b/x]a as a contracta. We shall say that an ob is in the normal form if there are no redexes having embeddings in a.

2. We write $A_1, \ldots, A_n \vdash A$ to denote that, from the sequents $A_1, \ldots, A_n$ may be inferred the sequent A, $n \geq 0$ (with $n = 0$ we have $\vdash A$, i.e., "the sequent A is provable"); $a \longleftrightarrow b$ denotes $\vdash a \rightarrow b$ in the calculus of λ-reduction.

3. Postulates of λ-reduction. α; β; $\rho: a \rightarrow a$; μ; ν; ξ; τ. The expression a red b denotes $\vdash a \rightarrow b$ in the calculus of λ-reduction.

Propositions. 1. If $a \longleftrightarrow b$, there exists an ob c such that a red c and b red c (Church-Rosser theorem).

2. If $ca_1 \ldots a_n \longleftrightarrow b$, where c is a variable or Ξ, there exist obs $b_1, \ldots, b_n$ such that $b \text{ red } cb_1 \ldots b_n$ and $a_i \text{ red } b_i$ $(i = 1, \ldots, n)$.

3. A λΞ-system and λΞ*-system are noncontradictory.

The proof of Proposition 3 is similar to the proof of the theorems on noncontradiction in [1,2] and is based on the Church-Rosser theorem.

In the formulated systems we can construct obs (operators) P, &, $\vee$, $\neg$, Π, $\exists$ having the properties of logical connectives and quantors, in the sense that the rules characterizing them, of introduction into the succeedent and antecedent, will be admissible [1-3]. For instance, for the operator $\neg \rightleftharpoons \lambda x(Px(\Pi I))$ may be constructed the inferences

$$a, \Gamma \Rightarrow \theta, x \vdash \Gamma \Rightarrow \theta, \neg a \quad (x \overline{\in} a, \Gamma, \theta);$$

$$\Gamma \Rightarrow \theta, a \vdash \neg a, \Gamma \Rightarrow \theta, b,$$

so that this operator can be interpreted as a logical connective of negation.

When defining new concepts it is convenient to use the abbreviations:

$$\lambda x \,.\, a \rightleftharpoons \lambda x a; \; \lambda x_1 \ldots x_n \,.\, a \rightleftharpoons \lambda x_1 (\lambda x_2 \ldots x_n \,.\, a), \; n > 1.$$

The principle of combinatory completeness: $(\lambda x_1 \ldots x_n \,.\, a)\, b_1 \ldots b_n \longleftrightarrow [b_1, \ldots, b_n / x_1, \ldots, x_n] a$, may then be proved for the operator of abstraction.

II. DEDUCTIVE SYSTEMS WITH OBS HAVING NORMAL FORMS

Definitions. 1. If a red b and the ob b are in the normal form, we shall say that ob a has the normal form, and b will be regarded as the normal form of a.

2. Deductive systems of λ-conversion and combinatory logic will be regarded as normal if *all* the objects, encountered in the postulates of the relevant extensions, have normal forms (see the Supplement to [2]).

For normal systems, we agree to write $\Vdash$ instead of $\vdash$.

Propositions. 1. Given any obs a and b having normal forms, $a \longleftrightarrow b$ or it is false that $a \longleftrightarrow b$.

2. The property "of having the normal form" is recursively undecidable (see [4,5] and the references to [1,2]).

3. There exist obs a and b such that their application (ab) has no normal form, whereas both a and b have normal forms. For instance, the ob λx.xx is in the normal form, but the ob $(\lambda x.xx)\lambda x.xx$ does not have normal form.

4. The ob Ξab has normal form if and only if a and b have normal forms.

5. The logical connectives and quantifiers designed in [1-3] have normal forms, and if all the obs of premises in the inferences characterizing these connectives and quantifiers have normal forms, then all the obs of concluding sequents will also have normal forms. However, the fact that all obs of concluding sequents have normal forms does not imply that all obs of premises have normal forms. For instance, the fact that (IIa) and b do not have normal form obviously does not always imply that (ab) has the normal form.

6. **Cut Theorem.** If $\Vdash \Delta \Rightarrow \Lambda, a$ and $\Vdash a, \Gamma \Rightarrow \theta$, then $\Vdash \Delta, \Gamma \Rightarrow \Lambda, \theta$.

It must be emphasized that the cut theorem only holds for normal system in the context of a division of obs into classes, closed with respect to conversion and substitution (only obs of one class participate in inferences); it was shown in [1-3] that, for systems with arbitrary objects, the cut rule $\frac{\Delta \Rightarrow \Lambda, a;\ a, \Gamma \Rightarrow \theta}{\Delta, \Gamma \Rightarrow \Lambda, \theta}$ is inadmissible.

We shall concentrate here on the expressive potentialities of the constructed systems, and postpone for the present the proof of the cut theorem. It will be borne in mind that the theorems on noncontradictoriness are retained for normal systems.

III. THE EQUALITY OPERATOR $Q \rightleftharpoons \lambda xy . \Xi(CIx)(CIy)$.

The following propositions characterize the operator Q as the symbol of equality.

1. $xa, \Gamma \Rightarrow \Lambda,\ xb \vdash \Gamma \Rightarrow \Lambda,\ Qab\ (x \overline{\in}\ a, b, \Gamma, \Lambda)$;
2. $\Delta \Rightarrow \Lambda, ca;\ cb, \Gamma \Rightarrow \theta \vdash Qab, \Delta, \Gamma \Rightarrow \Lambda, \theta$;
3. $a \rightarrow b \vdash Qab$; 4. $\vdash Qaa$; 5. $\vdash Qab \Rightarrow Qba$;
6. $\vdash \mathfrak{Q}(a_i b_i, n), ca_1 \ldots a_n \Rightarrow cb_1 \ldots b_n$;
7. $\vdash Qab, Qda \Rightarrow Qdb$;
8. $\vdash \mathfrak{Q}(a_i b_i, n). cb_1 \ldots b_n \Rightarrow ca_1 \ldots a_n$;
9. $\vdash \mathfrak{Q}(a_i b_i, n) \Rightarrow Q(ca_1 \ldots a_n)(cb_1 \ldots b_n)$;
10. $\vdash \mathfrak{Q}(a_i b_i, n) \Rightarrow Q(cb_1 \ldots b_n)(ca_1 \ldots a_n)$;

here, $\vdash \Gamma \rightleftharpoons \vdash \Rightarrow \Gamma$; $\mathfrak{Q}(a_i b_i; n) \rightleftharpoons Qa_1b_1, \ldots, Qa_nb_n$.

Propositions 1 and 2 provide the basis for the rules for introducing the equality operator; when proving them, we shall use the rules for Ξ and the rules of λ-cut. If all the obs of premises in Propositions 1 and 2 have normal forms, the obs of concluding sequents will also have normal forms (notice that the obs xa and ax have normal forms if and only if a has the normal form; see the Supplement to [2]).

Theorems 3-10 will be proved in systems with single-succeedent sequents. Propositions 3 and 4 will follow in accordance with the rules μ, of ascent (rule 1 of extension) and introducing Q, while the proof of 4 starts from the conversion $a \leftrightarrow a$. The proof of 5 is combinatorial: by virtue of the properties $S \rightleftharpoons \lambda xyz.xz(yz)$ and $K \rightleftharpoons \lambda xy.x$ we have $\vdash SQ(Ka)a$ and $\vdash SQ(Ka)b \Rightarrow Qba$, then we introduce Q on the left. The proof of 6 is by induction on n: with n = 1, from $ca \leftrightarrow ca$ and $cb \leftrightarrow cb$ we obtain $\vdash Qab, ca \Rightarrow cb$ is the Q principle; the inductive step utilizes the properties of the combinator $\lambda xy_1 \ldots y_k z.xzy_1 \ldots y_h$, $k>0$. Proposition 7 is a particular case of 6: n = 1 and $c \rightleftharpoons Qd$. The proof of 8 is based on 6 and 4 and the properties of S and K. Theorem 9 follows from 7 and the properties of the ob $B^n \rightleftharpoons \lambda xy_1 \ldots y_{n+1} . x(y_1 \ldots y_{n+1})$. Proposition 10 is not a consequence of 9 and 5 (the cut rule is inadmissible), and is proved independently, on the basis of 8 and B^n.

We shall agree to write $\dot{\vdash}$ in order to indicate that no obs having the normal form participate in the inference constructed by us.

11. $\dot{\vdash} \neg(Qab)$ for any objects a and b.

Proof. Let $x \overline{\in}\ a, b$. We define $R_x \rightleftharpoons \lambda y. P(yy)x$ and $L_x \rightleftharpoons R_x R_x$. We infer successively $L_x \leftrightarrow PL_x x$; $\vdash PL_x x$, $L_x \Rightarrow x$ is the P principle; $\vdash L_x, L_x \Rightarrow x$; $\vdash L_x \Rightarrow x$; $\vdash PL_x x$; $\vdash L_x$. By the λ-cut rule we obtain $\vdash KL_x a$ and $\vdash KL_x b \Rightarrow x$, whence, by introducing Q on the left and ¬ on the right: $\vdash Qab \Rightarrow x$ and $\vdash \neg(Qab)$.

Obviously, the fact that the sequents $\Rightarrow Qaa$ and $\Rightarrow \neg(Qab)$ can be proved simultaneously for any obs a and b, contradicts the intuitive properties of the equality. To overcome this difficulty, we impose in Section II an additional restriction on the objects of the deductive part of the theory: in the extension postulates we allow only obs which have normal forms. In particular, the ob R_x, appearing in the proof of Theorem 11, is in the normal form and is therefore admissible in normal systems, whereas the ob $L_x \rightleftharpoons R_x R_x$ does not have the normal form and is not an object of normal systems; see the Supplement to [2].

IV. SOME PROPERTIES OF THE OPERATORS P, &, ∨, ¬, Π, ∃

We first remark that it is possible to define the existence quantifier more simply than in [1]:

$\exists \rightleftharpoons \lambda x.\Xi(B(\Xi x)K)I$ (see [3]).

Propositions. 1. $\Gamma \to \theta,\ ab \vdash \Gamma \to \theta, \exists a$;

2. $ax, \Gamma \to \theta \vdash \exists a, \Gamma \to \theta$ (θ is a nonempty collection of obs, $x \overline{\in} a, \Gamma, \theta$).

Abbreviations: $[a] \rightleftharpoons a$; $[a_1 \ldots a_n] \rightleftharpoons a_1 [a_2 \ldots a_n]$, $n > 1$.

3. $\vdash Pa(Pba)$; 4. $\vdash P(Pab)(P(Pa(Pbc))(Pac))$;

5. $\vdash Pa(Pb(\& ab))$; 6. $\vdash P(\& ab)a$; 7. $\vdash P(\& ab)b$;

8. $\vdash Pa(\bigvee ab)$; 9. $\vdash Pb(\bigvee ab)$; 10. $\vdash P(Pac)(P(Pbc)(P(\bigvee ab)c))$;

11. $P(Pab)(P(Pa(\neg b))(\neg a))$; 12. $\vdash P(\neg a)(Pab)$; 13. $\overset{\circ}{\vdash} P[\neg\neg a]a$;

14. $P(\Pi \lambda x.a)[b/x]a$; 15. $\vdash P[b/x]a(\exists \lambda x.a)$.

Propositions 3-15 correspond to the schemes of axioms of Kleene's groups A1 and A2 [6]. Theorems 3-12, 14, and 15, are proved in systems with single-succeedent sequents. To prove Proposition 13, we turn to systems with two-succeedent sequents; here, instead of $\vdash$ we write $\overset{\circ}{\vdash}$ (the circle indicates results relevant to "classical" systems, when there is no expectation of being able to find an inference in which only single-succeedent sequents participate).

16. There exists an ob a, such that $\vdash a$ and $\overset{\circ}{\vdash} \neg a$. For instance, $a \rightleftharpoons \Xi II$, as distinct from the example quoted in [3]; the ob ΞII is in normal form; and the fact that $\overset{\circ}{\vdash} \neg(\Xi II)$ can be proved in the same way as Theorem 11 of III (we use the combinator I instead of K).

17. If $\vdash \Gamma \to \theta, Pab$, then $\vdash a, \Gamma \to \theta, b$.

18. If $\vdash \Gamma \to Pca$ and $x \overline{\in} c, \Gamma$, then $\vdash \Gamma \to Pc(\Pi \lambda x.a)$.

19. If $\vdash b, \Gamma \to Pca$ and $x \overline{\in} c, \Gamma$, then $\vdash \Gamma \to Pc(\Pi \lambda x.Pba)$.

20. If $\vdash \Gamma \to Pac$ and $x \overline{\in} c, \Gamma$, then $\vdash \Gamma \to P(\exists \lambda x.a)c$.

21. If $\vdash b, \Gamma \to Pac$ and $x \overline{\in} c, \Gamma$, then $\vdash \Gamma \to P(\exists \lambda x.\& ba)c$.

22. Modus ponens: if $\Vdash \Delta \to \Lambda, a$ and $\Vdash \Gamma \to \theta, Pab$, then $\Vdash \Delta, \Gamma \to \Lambda, \theta, b$

(in the context of a division of obs into classes).

The proof of Proposition 22 is based on the P principle and the cut theorem; it was shown in [1-3] that, for systems with arbitrary objects, the rule $\dfrac{\Delta \Rightarrow \Lambda, a;\ \Gamma \Rightarrow \theta, Pab}{\Delta, \Gamma \Rightarrow \Lambda, \theta, b}$ is inadmissible.

V. THE LOGIC OF PREDICATES IN DEDUCTIVE SYSTEMS

With the formulas of pure logic of predicates we associate certain obs of a deductive system. For this, we divide the system variables into two groups: for example, suppose that we put in one group the variables x_i with $i = 2k$, and in the other, those with $i = 2k - 1$, $k > 0$ (by construction, x_i is $|\square \ldots \square|$ (see [1,3]); we shall assume that the subscript i indicates the number of embeddings $\square$ in x_i). With predicate variables we associate elements of one of these groups, and with subject variables, elements of the other.

With the formula $R(\alpha_1, \ldots, \alpha_n)$, where R is a predicate variable and $\alpha_1, \ldots, \alpha_n$ are subject variables ($n \geq 0$), we associate the ob $R^\circ \alpha_1^\circ \ldots \alpha_n^\circ$ and successively, with the formulas $(A \supset B)$, $(A \& B)$, $(A \bigvee B)$, $\neg A$, $\forall x A$, $\exists x A$, the obs $PA^\circ B^\circ$, $\& A^\circ B^\circ$, $\bigvee A^\circ B^\circ$, $\neg A^\circ$, $\Pi \lambda x^\circ.A^\circ$, $\exists \lambda x^\circ.A^\circ$ (the circle indicates obs corresponding to expressions of the logic of predicates).

1. It is easily shown that all obs, corresponding to formulas of the pure logic of predicates, have normal forms.

2. Let A be a provable formula of pure logic of predicates. Then $\Vdash A°$. If there is a proof of formula A, in which the modus ponens rule does not participate, then $\vdash A°$.

The proof follows from Propositions 3-5, 18, 20, and 22 of the previous section.

3. Let $\Gamma\to\Delta$ be a provable sequent of pure logic of predicates: $\Gamma°, \Delta°$ are collections of obs, corresponding to formulas of Γ, Δ, and $\mathfrak{A}$ is the empty word or ΠI. Then $\Vdash \Gamma° \Rightarrow \Delta°, \mathfrak{A}$, and if there exists a proof of the sequent $\Gamma\to\Delta$, in which the cut rule does not participate, then $\vdash \Gamma° \Rightarrow \Delta°, \mathfrak{A}$.

The proof follows from the fact that the rules of introduction into the succeedent and antecedent are admissible for the operators P, $\&$, $\vee$, $\neg$, Π, $\exists$, and from the cut theorem.

4. Let $\mathfrak{M}$ be a class of obs, closed with respect to conversion and substitution. Then, given any a of $\mathfrak{M}$, there will not exist inferences $\Vdash \Rightarrow a$ and $\Vdash \Rightarrow \neg a$, composed of obs of class $\mathfrak{M}$.

VI. CONSTRUCTION OF A FORMAL ARITHMETIC

A. The arithmetic operations. Definitions: 1) the numerals $Z_0 \rightleftharpoons KI$; $Z_{n+1} \rightleftharpoons \sigma Z_n$, where $\sigma \rightleftharpoons \lambda xyz.y(xyz)$ is the succession combinator; 2) the sum combinator $[+] \rightleftharpoons \lambda xy.x\sigma y$; 3) the product combinator $[\cdot] \rightleftharpoons \lambda xy.x(y\sigma)Z_0$; 4) the predecessor $\pi \rightleftharpoons \lambda x.x\omega(KZ_0)Z_1$ where $\omega \rightleftharpoons \lambda x.D(\sigma(xZ_0))(xZ_0)$, $D \rightleftharpoons$ $\rightleftharpoons \lambda xyz.z(Ky)x$; 5) the combinator $\pi_0 \rightleftharpoons \lambda x.x\omega(KZ_0)Z_0$.

Abbreviation: $a^{(0)}b \rightleftharpoons b$; $a^{(n+1)}b \rightleftharpoons a(a^{(n)}b)$, $n \geqslant 0$.

Propositions: 1. $Z_n ab \leftrightarrow a^{(n)}b$; 2. $Z_n \sigma Z_0 \leftrightarrow Z_n$; 3. $[+]Z_0 b \leftrightarrow b$; 4. $[+](\sigma a)b \leftrightarrow \sigma([+]ab)$; 5. $[\cdot]Z_0 b \leftrightarrow Z_0$; 6. $[\cdot](\sigma a)b \leftrightarrow [+]b([\cdot]ab)$; 7. $DabZ_0 \leftrightarrow a$; 8. $Dab(\sigma c) \leftrightarrow b$; 9. $Dab(B(B(CCd)B)Z_0) \leftrightarrow ad$; 10. $Dab(C(B(C(B(B(\sigma e)K))d))g) \leftrightarrow bd$; 11. $\pi_0(\sigma a) \leftrightarrow \sigma(\pi_0 a)$; 12. $\pi(\sigma a) \leftrightarrow \pi_0 a$; 13. $\pi_0 Z_n \leftrightarrow Z_n$, $n \geqslant 0$; 14. $\pi Z_0 \leftrightarrow Z_0$; 15. $\pi Z_n \leftrightarrow Z_{n-1}$, $n > 0$.

Propositions 1-15 may be proved in calculi with the combinatorial completeness principle.

16. $\vdash \neg(QZ_0(\sigma a))$ is proved by introducing Q on the left from $\vdash DTxZ_0$ and $\vdash DTx(\sigma a) \Rightarrow x$, where $T \rightleftharpoons QKK$ and $x \,\overline{\in}\, a$; we obtain $\vdash QZ_0(\sigma a) \Rightarrow x$, then introduce $\neg$.

17. $\vdash Q(\pi_0 a)a \Rightarrow Q[\pi_0 \sigma a](\sigma a)$; we use the sequents $\vdash Bx\sigma a \Rightarrow x(\sigma a)$ and $\vdash x[\pi_0 \sigma a] \Rightarrow Bx\sigma(\pi_0 a)$, and the rules for introducing Q, $x \,\overline{\in}\, a$.

B. The arithmetic operator

$$N \rightleftharpoons \lambda x.\Xi(\lambda y.\&(yZ_0)(\Pi\lambda z.P(yz)[y\sigma z]))\lambda y.yx.$$

18. $\vdash Nb, aZ_0, \Pi\lambda x.P(ax)[a\sigma x] \Rightarrow ab$ is the principle of mathematical induction; it is obtained from the sequents $\vdash \Pi\lambda x.P(ax)[a\sigma x] \Rightarrow \Pi\lambda x.P(ax)[a\sigma x]$, $\vdash aZ_0 \Rightarrow aZ_0$, $\vdash (\lambda y.yb)a \Rightarrow ab\ (y \,\overline{\in}\, b)$; we use the rules $\Rightarrow\&$, $\Xi\Rightarrow$ (introductions of & on the right and Ξ on the left). 19. $\vdash NZ_0$. 20. $\vdash Na \Rightarrow N(\sigma a)$.

21. $\Gamma \Rightarrow aZ_0$; $\Delta, ax \Rightarrow a(\sigma x) \vdash Nb, \Gamma, \Delta \Rightarrow ab\ (x \,\overline{\in}\, a, \Delta)$.

When proving Theorems 18-21, we take account of the structure of the operator N.

22. $\vdash Na \Rightarrow Q[\pi\sigma a]a$. The proof is based on Propositions 12, 13, 17, and 21 ($\vdash UZ_0$; $\vdash Uz \Rightarrow U(\sigma z)$; $U \rightleftharpoons \lambda x.Q(\pi_0 x)x$).

23. $\vdash Nb \Rightarrow Qb[\pi\sigma b]$; the proof is similar to the proof of 22.

24. $\vdash Na, Nb, Q(\sigma a)(\sigma b) \Rightarrow Qab$; using Theorem 5 of III and 23, and introducing Q on the left, we obtain $\vdash U_a(\sigma b), Nb \Rightarrow Qab$; $U_a \rightleftharpoons \lambda x.Q(\pi x)a\ (x \,\overline{\in}\, a)$; we transform 22 to $\vdash Na \Rightarrow U_a(\sigma a)$; then we again introduce Q on the left.

Remark. The arithmetic operator may be defined in another way:

$$N^n \rightleftharpoons \lambda x.\Xi\widetilde{N}^n\lambda y.yx, \quad \text{where } \widetilde{N}^n \rightleftharpoons \lambda x.\&^{n+1}(xZ_0)\ldots(xZ_n), \quad n \geqslant 0;$$

$$\&^1 \rightleftharpoons I, \; \&^2 \rightleftharpoons \&, \; \&^{m+1} \rightleftharpoons \lambda x_1 \ldots x_{m+1}.\&(\&^m x_1 \ldots x_m)x_{m+1} \; (m > 1).$$

In this case: 25. $\vdash N^m Z_n\ (0 \leqslant n \leqslant m)$; 26. $\vdash N^n a, N^m b, Q(\sigma a)(\sigma b) \Rightarrow Qab\ (n \geqslant 0, m \geqslant 0)$.

C. With each provable formula of formal arithmetic (see e.g., [6]) we associate a deductive sequent. We associate one-to-one with the variables of the arithmetic the variables of the deductive systems. We associate the term 0 and the ob Z_0. With the terms $(s+r)$, $(s\cdot r)$, s' and formulas $(s=r)$, $(A\supset B)$, $(A\,\&\,B)$, $(A\vee B)$, $\neg A$ we associate consecutively the obs

$$[+]\underline{s}\underline{r},\ [\cdot]\underline{s}\underline{r},\ \sigma \underline{s},\ Q\underline{s}\underline{r},\ P\underline{A}\underline{B},\ \&\underline{A}\underline{B},\ \vee\underline{A}\underline{B},\ \neg\underline{A}$$

(the bar below indicates obs corresponding to constructions of the formal arithmetic).

Let D be a provable formula. Then, if D is an axiom of the groups A1 (with postulate 8^0 or 8^1), or A2, or 15-21 [6], then we associate D with the sequent $\Rightarrow\underline{D}$, and meantime, with subformulas of the type $\forall xH$ and $\exists xH$ we associate obs $\Pi\lambda x.\underline{H}$ and $\exists\lambda x.\underline{H}$; we proceed similarly if D is axiom 13 or 14, except that, instead of $\Rightarrow\underline{D}$, we take respectively $Nx\Rightarrow\underline{D}$ or $Na, Nb\Rightarrow\underline{D}$ (here, a and b are variables, see [6]).

Now let D be the direct consequence of formula F. Then, if Nx does not enter into the antecedent of the sequent $\mathfrak{A}\Rightarrow\underline{F}$, corresponding to F,

$$\underline{\forall x A(x)} \rightleftharpoons \Pi\lambda x.\underline{A(x)},\quad \underline{\exists x A(x)} \rightleftharpoons \exists\lambda x.\underline{A(x)}$$

(see Postulates 9 and 12 of [6]), and we associate D with $\mathfrak{A}\Rightarrow\underline{D}$; if Nx enters into $\mathfrak{A}$, then

$$\underline{\forall x A(x)} \rightleftharpoons \Pi\lambda x.P(Nx)\underline{A(x)},\quad \underline{\exists x A(x)} \rightleftharpoons \exists\lambda x.\&(Nx)\underline{A(x)}$$

and we associate D with $\mathfrak{B}\Rightarrow\underline{D}$, where $\mathfrak{B}$ is obtained from the collection $\mathfrak{A}$ by striking out Nx. If D is the direct consequence of formulas E and F, and $\mathfrak{A}\Rightarrow\underline{E}$ and $\mathfrak{B}\Rightarrow\underline{F}$ are the sequents corresponding to E and F, then we associate D with the sequent $\mathfrak{A}, \mathfrak{B}\Rightarrow\underline{D}$.

Propositions. 27. All obs of sequents associated with provable formulas of the formal arithmetic have normal forms. (This follows immediately from the construction of the relevant sequents.)

28. If A is a provable formula of the formal arithmetic, then $\vdash\mathfrak{A}\Rightarrow A$, where $\mathfrak{A}$ is a collection (possibly empty) of obs of the type Nx, and if there exists a proof of formula A, in which no part is played by the modus ponens rule, then $\vdash\mathfrak{A}\Rightarrow\underline{A}$; here, if Nx is a term of the collection $\mathfrak{A}$, then x will enter freely into $\underline{A}$.

The proof is based on the propositions of the present section and Section IV. Notice that, for the arithmetic system with Postulate 8^1, we can confine ourselves to single-succeedent sequents in the relevant theorems.

VII. FORMALIZATION OF SOME SET-THEORETIC CONCEPTS

Definitions. The ordered n-tuple $\langle a_1, \ldots, a_n\rangle \rightleftharpoons \lambda x.xa_1\ldots a_n$, $x\overline{\in} a_1, \ldots, a_n$ (usually, in axiomatic theories of sets, is defined by axioms of utilizing capacity and of the pair (see, eg.., [7-9], while in the λΞ-system the n-ple is introduced by devices of λ-conversion);

$$\pi_i^n \rightleftharpoons \lambda y.y(\lambda x_1\ldots x_n.x_i),\quad 0<i\leqslant n;$$

$$Q_1 \rightleftharpoons \lambda xy.\&(\Xi xy)(\Xi yx);\ \equiv\rightleftharpoons \lambda xy.\&(Pxy)(Pyx);\ \varnothing \rightleftharpoons \lambda x.\neg(Qxx).$$

Propositions: 1. $\pi_i^n\langle a_1, \ldots, a_n\rangle \leftrightarrow a_i$; 2. $\vdash Q\langle a_1, \ldots a_n\rangle\langle b_1, \ldots, b_n\rangle \Rightarrow Qa_ib_i$; 3. $\vdash \Omega(a_ib_i, n) \Rightarrow Q_1\langle a_1, \ldots, a_n\rangle \times \langle b_1, \ldots, b_n\rangle$;

4. Convolution theorem: $\vdash \exists\lambda y.\Pi\lambda x_1\ldots\Pi\lambda x_n.\equiv(\langle x_1, \ldots, x_n\rangle y)a$ (cf. [9], Proposition 4.4); 5. $\vdash \varnothing a \Rightarrow b$; 6. $\vdash Ua$, $U \rightleftharpoons \lambda x.Qxx$.

Further, let NBG* be a first order theory with equality and with variables for classes and for sets, which contains the following set-theoretic axioms: of capacity $(((X\subseteq Y)\,\&\,(Y\subseteq X))\supset(X=Y))$, where X, Y are variables for classes; axioms of the pair, the empty set, the existence of classes, union, the set of all subsets, separation, substitution, and infinity, identical in their statement with the corresponding axioms of [9]. Below we shall only consider set-theoretic constructions NBG*, generated from variables. With these constructions we associate certain obs of the λΞ-system (we indicate them by an asterisk).

Let the elements of one of the two groups of $\lambda\Xi$-system variables (see V) correspond to the variables for sets, and the elements of the other group to the variables for classes. With the ordered n-ple $<\alpha_1, \ldots, \alpha_n>$ of the NBG* theory we associate the ob $\langle\alpha_1^{\bullet}, \ldots, \alpha_n^{\bullet}\rangle$; with the formula $(\alpha = \beta)$ we associate the ob $Q\alpha^*\beta^*$; with the formula $\langle\alpha_1, \ldots, \alpha_n\rangle\varepsilon\beta$ the ob $\langle\alpha_1^{\bullet}, \ldots, \alpha_n^{\bullet}\rangle\beta^{\bullet}$ or $\beta^{\bullet}\alpha_1^{\bullet}\ldots\alpha_n^{\bullet}$, since $\langle\alpha_1^{\bullet}, \ldots, \alpha_n^{\bullet}\rangle\beta^{\bullet} \rightarrow \beta^{\bullet}\alpha_1^{\bullet}\ldots\alpha_n^{\bullet}$ (ε is the symbol of belonging); with the formula $(\alpha\varepsilon\beta)$, where α is not an ordered n-ple, the ob $\beta^*\alpha^*$ or $<\alpha^*>\beta^*$, and finally, with the formula $(\alpha\subseteq\beta)$, the ob $\Xi\alpha^*\beta^*$. In the case of other formulas, containing obs in the same way as in Section V.

Theorem. Let A be a provable formula of the NBG* theory, and let there exist a proof T of the formula A, in which all applications of the capacity axiom are exhausted by propositions regarding the uniqueness of certain constructions. Then $\Vdash A^*$, and if, in addition, no use is made in T of the modus ponens rule, then $\vdash A^*$.

The ideas developed in the present article are based on the deductive interpretation of logical connectives and quantifiers (see [1-3]) and on the separation of classes of objects having normal forms (see Supplement to [2]). In the proposed systems, the unrestricted principle of convolution is retained. Restrictions on the cut rule and the Church-Rosser theorem are utilized when proving propositions concerning the noncontradictory nature of the constructed systems.

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4 June 1973

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