DEDUCTIVE OPERATORS OF COMBINATORY LOGIC

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The present paper is a continuation of [1]; it is devoted to the construction and investigation of deductive systems of combinatory logic with single-succedent sequents, i.e., sequents of the form $\Gamma \Rightarrow a$, where a is an object (ob) of the system and Γ is a (possibly empty) set of obs. As will be shown, such systems may be assumed to be intuitionistic.

§1. Variant of a system of combinatory logic with the functionality operator. *Alphabet:* $KSF\Box|()\rightarrow\Rightarrow.$* The definitions of variables, obs, combinatory and deductive sequents, and axiom schemes 1-6 are given in [1, §1].

Rules of inference:

$$1.\ \frac{a\rightarrow b;\ b\rightarrow c}{a\rightarrow c};\quad 2.\ \frac{a\rightarrow b}{a\Rightarrow b};\quad 3.\ \frac{\Gamma\Rightarrow a}{b,\ \Gamma\Rightarrow a};$$

$$4.\ \frac{a,\ a,\ \Gamma\Rightarrow b}{a,\ \Gamma\Rightarrow b};\quad 5.\ \frac{\Gamma,\ a,\ b,\ \Delta\Rightarrow c}{\Gamma,\ b,\ a,\ \Delta\Rightarrow c};\quad 6.\ \frac{\Gamma\Rightarrow a;\ a\rightarrow b}{\Gamma\Rightarrow b};$$

$$7.\ \frac{a,\ \Gamma\Rightarrow c;\ b\rightarrow a}{b,\ \Gamma\Rightarrow c};\quad 8.\ \frac{ax,\ \Gamma\Rightarrow b(cx)}{\Gamma\Rightarrow Fabc};\quad 9.\ \frac{\Gamma\Rightarrow ad;\ b(cd),\ \Delta\Rightarrow e}{Fabc,\ \Gamma,\ \Delta\Rightarrow e}$$

(in rule 8 $x \overline{\in} a,\ b,\ c,\ \Gamma$).

The definitions, abbreviations, and notation adopted in [1] are retained. Obs will be denoted by the letters $a,\ldots,g$ of the Latin alphabet. Rule 1 and axiom schemes 1-6 define pure combinatory logic (see [1-3]). Rule 2 is considered the rule of ascent; rules 3-5 are antecedent structural rules; rules 6 and 7 are combinatory rules; rules 8 and 9 are the rules of introduction of functionality.

The system formulated above will be called an F*-system of combinatory logic. The obs set up from K and S are considered combinators; the definitions and basic properties of the combinators I, B, C, B^2, Ψ, W, Y, Φ are given in [1]; more detailed information on combinators is given in [2,3].

Propositions. It is not difficult to see that Theorems 1-5 of [1, §1] hold in an F*-system.

6. If $\Gamma\Rightarrow a$ is a provable sequent and $\mathfrak{A}$ is its proof, then $\mathfrak{A}$ includes a sequent $\Delta\Rightarrow b$, from which $\Gamma\Rightarrow a$ is obtained by structural and combinatory rules (consequently $a\leftrightarrow b$) and which, in turn, is obtained by one of the following methods:

(1) by the rule of ascent, in which case $\Delta \doteqdot c$ and $c\leftrightarrow b$;

(2) by the rule of introduction of F on the right, in which case $b \doteqdot Fcde$ and $[cx,\ \Delta\Rightarrow d(ex)]$, $x\overline{\in} c,\ d,\ e,\ \Delta$;

(3) by the rule of introduction of F on the left, in which case $\Delta \doteqdot Fcde,\ \Delta_1,\ \Delta_2$; $[\Delta_1\Rightarrow cg]$; $[d(eg),\ \Delta_2\Rightarrow b]$.

7. An F*-system of combinatory logic is consistent, since there exist unprovable sequents, for example, $\Gamma\Rightarrow$; $\Gamma\Rightarrow a_1,\ \ldots,\ a_n$ $(n>1)$; $K\Rightarrow S$ (see Proposition 1.7 of [1]).

8. The rule of cut 1.13 of [1] is not admissible in an F*-system, while the set Δ_1 is

*The arrows $=\Rightarrow \Rightarrow$ are identical in meaning.

empty, $\Delta_2 \mp b$.

9. We retain the formulation and validation of the F principle. The rule 1.14 with empty Δ and Δ_1, like the 2.5 and 3.7 which follow from it, is also inadmissible; it should be noted that in [1], when the Curry paradox was derived from the assumption of the rule 3.7, the latter was employed for the particular case in which the sets Γ, Γ_1, Δ and Δ_1 are empty.

10. $[\rightarrow Faal]$ (the functional character of the combinator I) follows from $[ax \rightarrow a(Ix)]$ by the rule used to introduce F, $x \bar{\in} a$.

11. $[\rightarrow Fa(Fba)K]$ (the functional character of the combinator K).

Proof. (1) $[ax \rightarrow a(Kxy)]$; $x, y \bar{\in} a, b$; (2) $[by, ax \rightarrow a(Kxy)]$ — (1), the rules of ascent and thinning; (3) $[\rightarrow Fa(Fba)K]$ — (2), the introduction of F.

12. $[\rightarrow F(Fa(Fbc))(F(Fab)(Fac))S]$ (the functional character of S).

Proof. (1) $[Faby, az \rightarrow b(yz)]$ —the principle of F; $x, y, z \bar{\in} a, b, c$; (2) $[c(xz(yz)) \rightarrow c(xz(yz))]$; (3) $[az \rightarrow az]$; (4) $[Fbc(xz), Faby, az \rightarrow c(xz(yz))]$ — (1), (2), the introduction of F; (5) $[Fa(Fbc)x, az, Faby, az \rightarrow c(xz(yz))]$ — (3), (4), the introduction of F; (6) $[az, Faby, Fa(Fbc)x \rightarrow c(Sxyz)]$ — (5), interchange, contraction, combinatory rule; (7) $[\rightarrow F(Fa(Fbc))(F(Fab)(Fac))S]$ — (6), the introduction of F.

13. $[\rightarrow F(Fa(Fab))(Fab)W]$ (the functional character of W). The proof is analogous to the proof of Propositions 10-12.

§2. **Ξ*- and PΠ*-systems of combinatory logic.** The definitions of the operators of formal implication, implication, conjunction, disjunction, universal quantifier, and negation which are given in [1] and based on the concept of functionality are preserved here. All the propositions in §2-6 of [1] are easily reformulated for the case of single-succedent sequents of an F*-system; their proofs, except for Theorem 5.4, can also be retained (with appropriate changes). Theorem 5.4 now takes the form: $\frac{a, \Gamma \Rightarrow c, \; b, \Delta \Rightarrow c}{\vee ab, \Gamma, \Delta \Rightarrow c}$.

Proof. (1) $\Gamma \rightarrow Pac$; (2) $\Delta \rightarrow Pbc$; (3) $\Gamma, \Delta \rightarrow \&(Pac)(Pbc)$ — (1), , (2), the introduction of &; (4) $\Gamma, \Delta \rightarrow \Phi\&(Pa)(Pb)c$ — (3), combinatory rule; (5) $[Ic \rightarrow c]$; (6) $\Xi(\Phi\&(Pa)(Pb))I, \Gamma, \Delta \rightarrow c$ — (4), (5), the introduction of Ξ; (7) $\vee ab, \Gamma, \Delta \rightarrow c$ — (6), 5.1 from [1], combinatory rule.

Concerning the existence quantifier, see: A. S. Kuzichev, Dokl. AN SSSR, vol. 212, no. 6, 1973.

As in [1], the operator Ξ and the operators P and Π can be taken as the basis for the construction of combinatory systems. The transformation of an F*-system into a Ξ*-system and a PΠ*-system and the justification of the equivalence of these three systems can be carried out in a manner analogous to §2 and §6 [1].

In the Ξ*- and PΠ*-systems the lemmas concerning "noncombinatory" objects (see [1], §1, Proposition 5) hold, where in the Ξ*-system d is Ξ or a variable, and in the PΠ*-system d is either P or Π or a variable. We can prove theorems analogous to Proposition 6 of §1 of the present paper and obtainable from it by replacing items (2), (3):

in the Ξ-system*

(2) by the rule of introduction of Ξ on the right, in which case $b \mp \Xi cd$; $[cx, \Delta \rightarrow dx]$, $x \bar{\in} c, d, \Delta$;

(3) by the rule of introduction of Ξ on the left, in which case $\Delta \mp \Xi cd, \Delta_1, \Delta_2$; $[\Delta_1 \rightarrow ce]$; and $[de, \Delta_2 \rightarrow b]$ for some ob e;

in the PΠ-system*

(2) by the rule of introduction of P on the right, in which case $b \mp Pcd$; $[c, \Delta \rightarrow d]$; (3) by the rule of introduction of P on the left, in which case $\Delta \mp Pcd, \Delta_1, \Delta_2$; $[\Delta_1 \rightarrow c]$ and $[d, \Delta_2 \rightarrow b]$; (4) by the rule of introduction of Π on the right, in which case $b \mp \Pi c$; $[\Delta \rightarrow cx]$, $x \bar{\in} c, \Delta]$; (5) by the rule of introduction of Π on the left, in which case $\Delta \mp \Pi c, \Delta_1$; $[cd, \Delta_1 \rightarrow b]$ for some ob d.

Propositions with the operators P, &, *and* V

1. $[\rightarrow Paa]$ is obtained from $[a \rightarrow a]$ by the introduction of P.

2. $[Pa(Pbc),\ a,\ b \rightarrow c]$. *Proof.* $[Pbc,\ b \rightarrow c]$ —the principle of P; $[a \rightarrow a]$; $[Pa(Pbc),\ a,\ b \rightarrow c]$ — introduction of P.

3. $[\rightarrow P(Pa(Pab))(Pab)]$. *Proof.* $[a \rightarrow a]$; $[Pab,\ a \rightarrow b]$; $[Pa(Pab),\ a,\ a \rightarrow b]$; $[\rightarrow P(Pa(Pab))(Pab)]$.

4. $[\rightarrow P(Pab)(P(Pa(Pbc))(Pac))]$.

Proof. $[Pab,\ a \rightarrow b]$; $[c \rightarrow c]$; $[Pbc,\ Pab,\ a \rightarrow c]$; $[a \rightarrow a]$; $[Pa(Pbc),\ a,\ Pab,\ a \rightarrow c]$; $[\rightarrow P(Pab)(P(Pa(Pbc))(Pac))]$.

5. $[a,\ b \rightarrow \&ab]$. 6. $[\&ab \rightarrow a]$. 7. $[\&ab \rightarrow b]$. 8. $[a \rightarrow \vee ab]$. 9. $[b \rightarrow \vee ab]$. Theorems 5-9 follow from $[a \rightarrow a]$ or $[b \rightarrow b]$ by the rules of introduction of conjunction or disjunction.

10. $\dfrac{a,\ b,\ \Gamma \Rightarrow c}{\&ab,\ \Gamma \Rightarrow c}$; in the proof we use the antecedent rules of the introduction of &, interchange, and contraction.

§3. Properties of the negation operator.

1. $\neg a \leftrightarrow Pa(\Pi I)$ (the combinatory characteristic of $\neg$).

2. $\dfrac{a,\ \Gamma \Rightarrow x}{\Gamma \Rightarrow \neg a}$ (the introduction of $\neg$ on the right; $x \bar{\in} a, \Gamma$).

Proof. $a, \Gamma \rightarrow x$; $a, \Gamma \rightarrow \Pi I$; $\Gamma \rightarrow Pa(\Pi I)$; $\Gamma \rightarrow \neg a$.

3. $\dfrac{\Gamma \Rightarrow a}{\neg a,\ \Gamma \Rightarrow b}$ (the introduction of $\neg$ on the left).

Proof. $[Ib \rightarrow b]$; $[\Pi I \rightarrow b]$; $\Gamma \rightarrow a$; $Pa(\Pi I),\ \Gamma \rightarrow b$; $\neg a,\ \Gamma \rightarrow b$.

4. $[\neg a,\ a \rightarrow b]$ is obtained from $[a \rightarrow a]$ by the introduction of $\neg$ into the antecedent.

5. $[a \rightarrow \neg(\neg a)]$. *Proof.* $[\neg a,\ a \rightarrow \Pi I]$; $[a \rightarrow P(\neg a)(\Pi I)]$; $[a \rightarrow \neg(\neg a)]$.

6. $[\rightarrow P(Pab)(P(Pa(\neg b))(\neg a))]$. *Proof.* $[a \rightarrow a]$; $[b,\ \neg b \rightarrow x]$, $x \bar{\in} a, b$; $[Pab,\ a,\ \neg b \rightarrow x]$; $[Pa(\neg b),\ a,\ Pab,\ a \rightarrow x]$; $[Pa(\neg b),\ Pab \rightarrow \neg a]$; $[\rightarrow P(Pab)(P(Pa(\neg b))(\neg a))]$.

7. We shall show that in combinatory systems the rule

$$\frac{[a,\ \Gamma \Rightarrow b];\ [a,\ \Gamma \Rightarrow \neg b]}{[\Gamma \Rightarrow \neg a]} \quad \left(\begin{array}{l}\text{sequential variant of}\\ \text{“reductio ad absurdum”}\end{array}\right)$$

is inadmissible.

Suppose that Rule 7 is admissible. We set $L \rightleftharpoons Y(CP(\Pi I))$, where Y is the paradoxical combinator [1,2]. We successively derive

$$L \leftrightarrow PL(\Pi I);\ L \leftrightarrow \neg L;\ [\neg L,\ L \rightarrow \Pi I];\ [L,\ L \rightarrow \Pi I];\ [L \rightarrow \Pi I];$$
$$[\rightarrow PL(\Pi I)];\ [\rightarrow \neg L];\ [\rightarrow L];\ [a \rightarrow \neg L];\ [a \rightarrow L].$$

From the last two sequents, by Rule 7, we obtain a contradiction (with respect to negation): $[\rightarrow \neg a]$ for any ob a.

8. $\dfrac{\Gamma \Rightarrow a}{\Gamma \Rightarrow \neg(\neg a)}$ (introduction of double negation on the right).

Proof. $\Gamma \rightarrow a$; $\neg a,\ \Gamma \rightarrow x$; $x \bar{\in} a, \Gamma$; $\Gamma \rightarrow \neg(\neg a)$.

9. $\dfrac{a,\ \Gamma \Rightarrow x}{\neg(\neg a),\ \Gamma \Rightarrow b}$ (introduction of double negation on the left, $x \bar{\in} a, \Gamma$). The proof is obtained by successive introduction of $\neg$ into the antecedent and into the succedent.

10. $[Pab \rightarrow P(\neg b)(\neg a)]$. *Proof.* $[a \rightarrow a]$; $[b,\ \neg b \rightarrow x]$, $x \bar{\in} a, b$; $[Pab,\ a,\ \neg b \rightarrow x]$; $[\neg b,\ Pab \rightarrow \neg a]$; $[Pab \rightarrow P(\neg b)(\neg a)]$.

11. $[Pa(\neg b) \rightarrow Pb(\neg a)]$ is proved in a manner analogous to 10.

12. $[Pa(Pbc), \neg(\neg a), \neg(\neg b) \Rightarrow \neg(\neg c)]$. *Proof*: $[c, \neg c \Rightarrow x]$, $x \bar{\in} a, b, c$; $[b \Rightarrow b]$; $[Pbc, b, \neg c \Rightarrow x]$; $[a \Rightarrow a]$; $[Pa(Pbc), a, b, \neg c \Rightarrow x]$; $[Pa(Pbc), \neg(\neg a), \neg(\neg b), \neg c \Rightarrow x]$; $[Pa(Pbc), \neg(\neg a), \neg(\neg b) \Rightarrow \neg(\neg c)]$. The last two sequents are obtained by the rules of introduction of negation: the first by the introduction of $\neg\neg$ in the antecedent, the second by the introduction of $\neg$ on the right.

In what follows it will be useful to introduce one more *contraction* of parentheses:

$$[a] \rightleftharpoons a; \ [ab] \rightleftharpoons (ab); \ [a_1 \ldots a_n] \rightleftharpoons (a_1 [a_2 \ldots a_n]), \quad n > 2.$$

13. $[[\neg\neg(Pab)] \Rightarrow P[\neg\neg a][\neg\neg b]]$. *Proof:* $[a \Rightarrow a]$ $[b, \neg b \Rightarrow x]$, $x \bar{\in} a, b$; $[Pab, a, \neg b \Rightarrow x]$; $[[\neg\neg(Pab)], [\neg\neg a] \Rightarrow [\neg\neg b]]$.

14. $[\neg(\neg(Pab)), \neg(\neg(Pbc)) \Rightarrow \neg(\neg(Pac))]$. *Proof:* $[Pab, a \Rightarrow b]$; $[c \Rightarrow c]$; $[Pbc, Pab, a \Rightarrow c]$; $[Pbc, Pab \Rightarrow Pac]$; $[\neg(Pac), Pbc, Pab \Rightarrow x]$, $x \bar{\in} a, b, c$; $[\neg(\neg(Pab)), \neg(\neg(Pbc)) \Rightarrow \neg(\neg(Pac))]$.

15. $[\neg(\neg(\&ab)) \Rightarrow \&[\neg\neg a][\neg\neg b]]$. *Proof:* $[a, \neg a \Rightarrow x]$, $x \bar{\in} a, b$; $[\&ab, \neg a \Rightarrow x]$ —introduction of &; $[\neg(\neg(\&ab)), \neg a \Rightarrow x]$; $[\neg(\neg(\&ab)) \Rightarrow \neg(\neg a)]$; in an analogous manner we construct $[\neg(\neg(\&ab)) \Rightarrow \neg(\neg b)]$; $[\neg(\neg(\&ab)) \Rightarrow \&[\neg\neg a][\neg\neg b]]$ —introduction of & in the succedent.

16. $[\&[\neg\neg a][\neg\neg b] \Rightarrow \neg(\neg(\&ab))]$. *Proof:* $[a, b \Rightarrow \&ab]$; $[a, b, \neg(\&ab) \Rightarrow x]$, $x \bar{\in} a, b$; $[\&[\neg\neg a][\neg\neg b] \Rightarrow \neg(\neg(\&ab))]$.

17. $[[\neg\neg\neg a] \Rightarrow \neg a]$. *Proof:* $[a, \neg a \Rightarrow x]$, $x \bar{\in} a$; $[a, [\neg\neg\neg a] \Rightarrow x]$; $[[\neg\neg\neg a] \Rightarrow \neg a]$.

18. $[\Rightarrow \neg(\&a(\neg a))]$ (negation of contradiction).

Proof: $[a, \neg a \Rightarrow x]$, $x \bar{\in} a$; $[\&a(\neg a) \Rightarrow x]$; $[\Rightarrow \neg(\&a(\neg a))]$.

19. $[\neg(\vee ab) \Rightarrow \&(\neg a)(\neg b)]$. *Proof:* $[a \Rightarrow \vee ab]$; $[\neg(\vee ab), a \Rightarrow x]$, $x \bar{\in} a, b$; $[\neg(\vee ab) \Rightarrow \neg a]$; in an analogous manner we derive $[\neg(\vee ab) \Rightarrow \neg b]$; by the rule of introduction of & we obtain the required sequent.

20. $[\&(\neg a)(\neg b) \Rightarrow \neg(\vee ab)]$. *Proof:* $[\&(\neg a)(\neg b), a \Rightarrow x]$, $x \bar{\in} a, b$; $[\&(\neg a)(\neg b), b \Rightarrow x]$; $[\vee ab, \&(\neg a)(\neg b) \Rightarrow x]$; $[\&(\neg a)(\neg b) \Rightarrow \neg(\vee ab)]$.

21. $[\vee(\neg a)b \Rightarrow Pab]$. *Proof:* $[\neg a, a \Rightarrow b]$; $[b \Rightarrow b]$; $[\vee(\neg a)b, a \Rightarrow b]$; $[\vee(\neg a)b \Rightarrow Pab]$.

22. $[\Rightarrow [\neg\neg(\vee a(\neg a))]]$. *Proof:* $[a \Rightarrow \vee a(\neg a)]$; $[\Pi I \Rightarrow x]$, $x \bar{\in} a$; $[P(\vee a(\neg a))(\Pi I), a \Rightarrow x]$; $[\neg(\vee a(\neg a)) \Rightarrow \neg a]$; $[\neg(\vee a(\neg a)) \Rightarrow \vee a(\neg a)]$; $[P(\vee a(\neg a))(\Pi I), \neg(\vee a(\neg a)) \Rightarrow \Pi I]$; $[\neg(\vee a(\neg a)) \Rightarrow \Pi I]$; $[\Rightarrow [\neg\neg(\vee a)(\neg a)]]$.

23. $[\Rightarrow \neg(\neg(P[\neg\neg a]a))]$. *Proof:* $[a \Rightarrow P[\neg\neg a]a]$; $[\Pi I \Rightarrow x]$, $x \bar{\in} a$; $[P(P[\neg\neg a]a)(\Pi I), a \Rightarrow x]$; $[\neg(P[\neg\neg a]a) \Rightarrow \neg a]$; $[\Pi I \Rightarrow a]$; $[P(\neg a)(\Pi I), \neg(P[\neg\neg a]a) \Rightarrow a]$; $[\neg(P[\neg\neg a]a) \Rightarrow P[\neg\neg a]a]$; $[\Pi I \Rightarrow \Pi I]$; $[P(P[\neg\neg a]a)(\Pi I), \neg(P[\neg\neg a]a) \Rightarrow \Pi I]$; $[\neg(P[\neg\neg a]a) \Rightarrow \Pi I]$; $[\Rightarrow \neg(\neg(P[\neg\neg a]a))]$.

§4. *Some propositions with two-succedent sequents.* In the present section we shall consider systems of combinatory logic with deductive sequents whose right side consists of no more than two terms (obs). The construction of the corresponding F**-, Ξ**-, and PΠ**-systems can be carried out analogously to §1, 2, and 6 [1].

1. $[\Rightarrow P(P(Pab)a)a]$. *Proof:* $[a \Rightarrow a, b]$; $[\Rightarrow a, Pab]$; $[a \Rightarrow a]$; $[P(Pab)a \Rightarrow a]$; $[\Rightarrow P(P(Pab)a)a]$.

2. $[\Rightarrow a, \neg a]$. *Proof:* $[a \Rightarrow a, \Pi I]$; $[\Rightarrow a, Pa(\Pi I)]$; $[\Rightarrow a, \neg a]$.

3. $[\neg(\neg a) \Rightarrow a]$. *Proof:* $[\Rightarrow a, \neg a]$; $[\Pi I \Rightarrow a]$; $[P(\neg a)(\Pi I) \Rightarrow a]$; $[\neg(\neg a) \Rightarrow a]$.

4. $[P(\neg a)b \Rightarrow P(\neg b)a]$. *Proof:* $[\Rightarrow a, \neg a]$; $[b, \neg b \Rightarrow a]$, $[P(\neg a)b, \neg b \Rightarrow a]$; $[P(\neg a)b \Rightarrow P(\neg b)a]$.

5. $[P(\neg a)(\neg b) \Rightarrow Pba]$ is proved in a manner analogous to 4.

6. $[\Rightarrow \vee a(\neg a)]$. *Proof:* $[\Rightarrow a, \vee a(\neg a)]$; $[\Rightarrow \vee a(\neg a), a]$; $[\Rightarrow \vee a(\neg a), \vee a(\neg a)]$; $[\Rightarrow \vee a(\neg a)]$.

7. $[Pab \Rightarrow \vee(\neg a)b]$. *Proof:* $[\Rightarrow \vee(\neg a)b, a]$; $[b \Rightarrow b]$; $[b \Rightarrow \vee(\neg a)b]$; $[Pab \Rightarrow \vee(\neg a)b, \vee(\neg a)b]$; $[Pab \Rightarrow \vee(\neg a)b]$.

8. $[\neg(\&ab) \Rightarrow \vee(\neg a)(\neg b)]$. *Proof:* $[\Rightarrow \vee(\neg a)(\neg b), a]$; $[\Rightarrow \vee(\neg a)(\neg b), b]$; $[\Rightarrow \vee(\neg a)(\neg b), \&ab]$; $[\neg(\&ab) \Rightarrow \vee(\neg a)(\neg b)]$.

Conclusion. 1. A. A. Markov, in his On the Logic of Constructive Mathematics [4], gives various treatments of the concept of implication, including deductive implication. In [1] and the present paper we have concerned ourselves with the construction and investigation of the theory of deductive implication in combinatory logic; we have extended the deductive concept of implication to the operators of functionality and formal implication and to the universal quantifier. Treating implication as deductive naturally leads to the construction of a hierarchy of corresponding languages. Multistage theories are constructed one after another by successive extensions based on the rule of introduction of implication on the right ("the deduction theorem"). The corresponding deductive constructions of combinatory logic have been investigated in our earlier studies published in Vestnik Mosk. un-ta, matem., mekhan., no. 3, pp. 50-55, 1970; no. 4, pp. 69-74, 1971; no. 1, pp. 37-43, 1972; in Dokl. AN SSSR, vol. 198, no. 4, pp. 759-761, 1971; and in the collection Combinatory Analysis, No. 1, Izd-vo MGU, pp. 105-119, 1971.*

In the present paper the concept of deductive implication is connected not only with the rule of its introduction on the right but also with introduction on the left. This has led us to the construction of two-level theories. The first level is formed by pure combinatory logic in the sequential variant; the second level (the top of the hierarchy) is constructed as a deductive extention of the first on the basis of Gentzen's theory of sequents. We have investigated the means of expression of the proposed systems of combinatory logic; we have shown that the language of each of the constructions is closed with respect to all logical connectives. The results of §1-3 of this paper are intuitionistic in character. In this connection the theorems of greatest interest are those on the properties of the negation operator (§3); in Theorems 1-8 (§4) we prove some laws of the classical logic of propositions. Making use of the Church-Rosser theorem,** we show that each of the systems is consistent.

2. The properties of the paradoxical combinator Y were used above for constructing an object L such that $(\rightarrow L)$ and $(\rightarrow \neg L)$. The presence of the object L brings the systems considered closer to certain variants of logic described by Fitch (see the article "A method for avoiding the Curry paradox" [7] and other papers by Fitch regularly published in the journal Symbolic Logic, as well as in D. Prawitz's book [8]).

3. Making use of the combinators K, S, and I in pure combinatory logic with respect to an arbitrary object a we can construct a new object $\{x_1 \ldots x_n\}a$, where $\{x_1 \ldots x_n\}$ appears in the role of an abstraction operator with respect to the variables $x_1, \ldots, x_n$, $n > 0$. For the abstraction operator we can prove the principle of combinator completeness:

$$(\{x_1 \ldots x_n\}a)b_1 \ldots b_n \leftarrow [b_1, \ldots, b_n/x_1, \ldots, x_n]a,$$

where $[b_1, \ldots, b_n/x_1, \ldots, x_n]a$ is the result obtained by substituting the obs $b_1, \ldots,$ b_n simultaneously into the ob a at the points where the corresponding graphically different variables $x_1, \ldots, x_n$ $n > 0$ appear.

The concept of an abstraction forms the nucleus of Church's theory of λ-conversion [5]. The property of combinator completeness in the calculi of λ-conversion is given, as a rule, by the axiom scheme (β):

$$(\lambda x.a)\, b \text{ conv } [b/x]\, a.$$

The calculi of λ-conversion can be used as the basis for deductive extensions in which the properties of the logical connectives can be expressed. Systems of λ-conversion with operators of functionality, formal implication, implication, and the universal quantifier can be constructed analogously with the corresponding extensions of the theory of combinators.

4. Special attention should be given to the concept of equality in deductive systems, since the most widespread definition of equality Q in terms of the operator Ξ and the

*See also Dokl. AN SSSR, vol. 209, no. 3, pp. 541-543, 1973; RZh Matematika, vol. 10, ref. A69, 1972; collection History and Methodology of the Natural Sciences, No. XIV, Izd-vo MGU, pp. 131-141, 1973.

**The Church-Rosser theorem is one of the central theorems in combinatorially complete theories (see [2,3,5,6]). In [3] the Church-Rosser theorem is proved by a new and simpler method based on the ideas of P. Martin-Löf and W. Tait.

combinators Ψ, C, and I (see, for example, [2,3]) yields $[\rightarrow \neg (Qab)]$ for any obs a and b, which obviously contradicts the intuitive properties of equality. In the proposed theories, naturally, the role of equality can be played by the conversion transformation. It would be useful to investigate what arithmetic and set-theoretic facts can be proved with this.

The introduction of an operator characterizing conversion appears insufficient to express all the properties of equality. Therefore we propose placing additional restrictions on the rules in the deductive part of the theory which prohibit the simultaneous proof of the sequents $\rightarrow L$ and $\rightarrow \neg L$ (see Item 2 of the Conclusion). These restrictions relate primarily to the rules of contraction in the antecedent and the introduction of implication on the left. The formulation and detailed investigation of combinatorially complete deductive systems with these restrictions can form the subject of a special article. Here we note only that in the proposed variant of the theory we can preserve the object Q as the operator of equality, prove once again the above theorems on the properties of logical connectives, introduce the concept of an arithmetic operator, and formulate and prove the fundamental postulates of arithmetic and some set-theoretic propositions.

5. In evaluating the possibilities of the proposed theories, we should emphasize that, firstly, in calculi with the principle of combinatory completeness all partial recursive functions are representable (see, for example, [3]); secondly, the property of combinatory completeness for n = 1 is the analog of the unrestricted set-theoretic principle of comprehension (for the principle of comprehension (selection), see, for example, [9]); thirdly, the calculi of λ-conversion and combinatory logic are being more and more widely applied in the theory of programming languages [3,10,11].

SUPPLEMENT TO THE ARTICLE "DEDUCTIVE OPERATORS OF COMBINATORY LOGIC"

1. In the proposed deductive extensions of pure combinatory logic and the theory of λ-conversion, the properties of the connectives of the calculus of predicates in combination with the principle of combinatory completeness make it possible to construct an object a for which the sequents $\rightarrow a$ and $\rightarrow \neg a$ will be provable individually without requiring direct recourse to objects which have no normal form (see Item 2 of the Conclusion); for example,

$$a \rightleftharpoons \bar{\exists} (\lambda z.\ \Pi \lambda y.\ \equiv (zy) [\neg yy]),$$

where $\equiv \rightleftharpoons \lambda xy.\ \&(Pxy)(Pyx)$, $\exists \rightleftharpoons \lambda x. \Xi (B(\Xi x)K)I$; in the proof of the first sequent we find the ob $T \rightleftharpoons \lambda x.\ \neg(xx)$, which has a normal form and is such that its application to itself has no normal form. The construction of similar examples in systems with the unrestricted set-theoretic principle of comprehension is taken from the works of W. Ackermann and K. Schütte (our attention was drawn to this by A. G. Dragalin).

2. Let us formulate a variant of the restriction on the rules in the deductive part of the theory (see Item 4 of the Conclusion).

An ob a will be called normal if the ob and all its components (subobs) have normal forms. Deductive systems are considered normal if *all* objects of the deductive part (obs of the second level) are normal.

The class of all normal obs is not closed with respect to the operation of substitution: there exist normal obs a and b such that the ob [b/x]a is not normal, for example, $a \rightleftharpoons Tx$, $b \rightleftharpoons T$.

With each normal ob a we naturally associate some region of variation of the variable x with respect to a. A normal ob b is considered to belong to the region $\mathfrak{A}$ of variation of x with respect to a if the ob [b/x]a is normal; this region will also be denoted by

$\mathfrak{A}^x$, $\mathfrak{A}_a$ or $\mathfrak{A}_a^x$.

Let A be a set of sequents and let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be the regions of variation of x with respect to the obs which are the components of the sequents of the set A. Then we say that the region $\mathfrak{A}$ of variation of x with respect to A is the intersection of the regions $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, i.e., $\mathfrak{A} \rightleftharpoons \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n$.

It is obvious that for any normal obs a and b, the classes $\mathfrak{A}_{ax}^x$ and $\mathfrak{A}_{yb}^y$ $(x \bar{\in} a,\ y \bar{\in} b)$ contain b and a, respectively, if and only if the ob ab is normal.

With the regions of variation of the variables we also naturally associate the regions of action of the operators obtained by the rules of their introduction. Thus, the region of action of the operator Ξ in the sequent $\Gamma \Rightarrow \Delta, \Xi ab$, obtained from the sequent $ax, \Gamma \Rightarrow \Delta, bx$ $(x \overline{\in} a, b, \Gamma, \Delta)$, coincides with the region of variation of x relating to the last sequent, and the region of action of Ξ in the sequent $\Xi ab, \Delta, \Gamma \Rightarrow \Lambda, \theta$, obtained by the rule of introduction of Ξ on the left is the intersection of the regions $\mathfrak{A}$ and $\mathfrak{B}$, which relate, respectively, to the sequents $\Delta \Rightarrow \Lambda, ac$ and $bc, \Gamma \Rightarrow \theta$ and contain the ob c which appears as a concrete representative of the region $\mathfrak{A} \cap \mathfrak{B}$.

Variables whose regions of variation in the above context do not coincide are considered different and are denoted by different letters. In particular, if it is known that the ob a belongs to the region $\mathfrak{A}$ and does not belong to the region $\mathfrak{B}$, then the variables corresponding to these regions are different and are denoted in the context under consideration by different letters (for example, x and y), and furthermore, if these regions become the regions of action of an operator (say, formal implication), then the operators are also naturally to be considered different (for example, instead of Ξ we can write $\Xi_{\mathfrak{A}}$ and $\Xi_{\mathfrak{B}}$ or Ξ_x and Ξ_y; frequently this distinction in notation follows from the difference between the variables of the abstraction operator: $\Xi\lambda x.b$ and $\Xi\lambda y.b$).

If we are given the proof of some sequent in a system with arbitrary objects and if in this proof we do not make use of objects which have no normal form, then in the corresponding normal system the proof can be completely preserved if we define the regions of variation of the variables and the regions of action of the operators which appear in the sequents of the proof—naturally, variables which are different (in the sense of normal systems) are now to be denoted differently, and the operators are differentiated accordingly. For example, using the ob T, in normal systems we prove the sequents

$$\Rightarrow \Xi\, \lambda z.\, \Pi \lambda y. \equiv (zy) [\neg\, yy] \text{ и } \Rightarrow \neg (\exists\, \lambda t.\, \Pi \lambda y. \equiv (ty) [\neg\, yy]);$$

in the proofs the regions of variation of the variables z and t (and consequently the regions of action of the related existence quantifiers) are different: the first includes the ob T, while the second does not.

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