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Equating for Recurrent Terms of λ-calculus and Combinatory Logic

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EQUATING FOR RECURRENT TERMS OF λ-CALCULUS AND COMBINATORY LOGIC

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ABSTRACT – In this paper equating problems for $\lambda\beta\eta$ calculus and Combinatory Logic are considered.

The main result is that:

1 - closed recurrent terms of order *0* can be all equated without equating to them any other closed term;

2 - every closed recurrent term of order *0* can be equated to an arbitrary closed term.

The same equating problems are taken into account for some other sets of terms which properly contain the set of recurrent terms of order *0*.

1. Introduction

We consider $\lambda\beta\eta$-calculus and Combinatory Logics as equality theories where two convertible or equal terms are names of the same object. In order to extend those theories by consistently adding some equalities, we notice that a term T of order *0* behaves as a free variable while reducing every context where T occurs, if we do not take into account the reductions performed from T.

Nevertheless, what we just asserted is not true from the point of view of equality: in fact for instance $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$ is such that $\Omega_3(\lambda x.xxx) = \Omega_3$, so that we can say that Ω_3 erases $\lambda x.xxx$ by conversion.

Thus, in order to find a set of objects for which equating is, so to say, a highly successful operation, since they have quite few characteristic properties, we interested on terms where reduction and equality in some sense are the same. These terms are the recurrent terms defined in [8] and having a subset $\mathcal{R}_0$ of closed terms of order *0*, we name *0*-recurrent (zero-recurrent), whose elements can be all consistently equated and moreover every element of $\mathcal{R}_0$ can be consistently equated to every closed term.

It is well known that all unsolvable terms can be consistently equated, but an arbitrary unsolvable term cannot be consistently equated to an arbitrary closed one. Moreover, the set $\mathcal{U}$ of all unsolvable terms is not recursively enumerable, hence the $\lambda\mathcal{U}$-calculus, which could be obtained by equating all elements of $\mathcal{U}$, is not effective. On the contrary, since to be a 0-recurrent term is a recursively enumerable relation, as it can be seen from [8], where a strategy h^+ distinguishes, in Weak Combinatory Logic, 0-recurrent terms, i.e. terms of the form $RU_1 \ldots U_h$, $h \geqslant 0$, with R and U_i recurrent, also from recurrent terms not in $\mathcal{R}_0$ (which are in h.n.f., hence not of order 0). Thus h^+ distinguishes $\mathcal{R}_0$ among the other sets of terms it recognizes(1).

By taking into account that recurrent terms not in $\mathcal{R}_0$ of $\lambda\beta\eta$ are not in h.n.f. but that nevertheless they are not of order 0, it is possible to traslate h^+ into a strategy for $\lambda\beta\eta$ which reveals also $\mathcal{R}_0$ of $\lambda\beta\eta$ [9]. Then the $\lambda\mathcal{R}_0$-calculus which could be obtained by introducing a new symbol to which every element of $\mathcal{R}_0$ is reducible, should be effective, because it can be verified in finite time whether two "$\lambda\mathcal{R}_0$ terms" are convertible.

2. Notations and preliminaries

We suppose the reader acquainted with λ-calculus, Combinatory Logic (CL) and Weak CL (WCL). By $\lambda\beta\eta \vdash$, CL $\vdash$, provability in $\lambda\beta\eta$, in CL is denoted, as usual. Λ stands for the set of λ-terms, Λ_0 for the set of all closed λ-terms, C_0 for the set of all combinators, WC_0 for that of combinators of WCL.

We use x, y, z for variables; R for a redex; $F, G, M, N, P, Q, T, U, V$ for terms in Λ or in Λ_0, all of them somewhere also with subscripts. The symbol $\longrightarrow$ stands for contraction, $\twoheadrightarrow$ for reduction and $=$ for conversion. $T^{(n)}$, $n \geqslant 0$, is a term obtained from T by means of n contractions. $C[\ ,\ldots,\ ]$ is a context with $n \geqslant 1$ holes.

The following definitions of sets of terms and their main properties are contained in [8] for WCL, and in [9] are extended to $\lambda\beta\eta$.

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(1) In [3] an algorithm, which uses Gross-strategy, has been subsequently found for $\lambda\beta$ which reveals the set of recurrent terms, named minimal forms, but not $\mathcal{R}_0$.

DEFINITION 1 - T is *recurrent* (r.) when and only when if $T \twoheadrightarrow P$ then $P \twoheadrightarrow T$.

For instance $\Omega_2 \equiv (\lambda x.xx)(\lambda x.xx)$, $\lambda y.\Omega_2$, $\lambda z.z\Omega_2$, $\Omega_2(\lambda y.y)$, and $z\Omega_2\Omega_2$ are *r*, while $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$ is not.

1 - If $U = T$ with T *r*. then $U \twoheadrightarrow T$.

In particular:

2 - If $T_1 = T_2$ with T_1, T_2 both r. then $T_1 \twoheadrightarrow T_2$ and $T_2 \twoheadrightarrow T_1$.

DEFINITION 2 - T is *head-recurrent* (h.r.) iff its head redex, if any, is r.

For instance $\Omega_2\Omega_3$, $\lambda xy.\Omega_2\Omega_3$, Ω_2, $\lambda y.\Omega_2$, $\Omega_2(\lambda y.y)$ are h.r., while $\Omega_3\Omega_2$ and $(\lambda xy.xx\Omega_3)(\lambda xy.xx\Omega_3)\Omega_3$ are not.

1 - If $U = T$, with $T \equiv \lambda x_1 \ldots x_m.\ RU_1 \ldots U_h$ $(m,h \geqslant 0)$ h.r., then $U \twoheadrightarrow \lambda x_1 \ldots x_m.RU_1' \ldots U_h'$, where either $U_i \equiv U_i'$ or $U_i \twoheadrightarrow U_i'$.

2 - If T is r. then T reduces to an h.r. term.

The set of r. terms of WCL turns out to be properly contained in that of h.r. ones, since a term in h.n.f. is considered, in [8], as a special h.r. term having a constant head in place of a r. head redex. Hence an h.r. term is seen as a kind of weak h.n.f..

For the purposes of this paper we restrict to r. and h.r. closed terms with no h.n.f. and we shall see later that those of them which are of order *0* are successfully equable in a specified sense. H.r. terms of order *0* can be obtained by applying a r.R to finitely many arbitrary terms and also by applying a r. term of order *0* to a finite number of arbitrary terms.

The rest of this paragraph is a generalization of what is contained in [6] for CL.

Let's now restrict to Λ_0 and let E be a subset of Λ_0.

DEFINITION 3 - $M \overset{E}{\text{——}} N$ iff there exist U and V both in E and an arbitrary closed F such that $M = FU$ and $N = FV$.

Let $\overset{E}{\sim}$ be the transitive closure of $\overset{E}{\text{——}}$. For $\overset{E}{\sim}$ it holds:

1. $M \overset{E}{\sim} M$

2. $M \overset{E}{\sim} N$ iff $N \overset{E}{\sim} M$

3. If $M \overset{E}{\sim} N$ and $N \overset{E}{\sim} P$ then $M \overset{E}{\sim} P$

4. If $M \overset{E}{\sim} N$ and $P \overset{E}{\sim} Q$ then $MP \overset{E}{\sim} NQ$

5. If M and N are in E then $M \overset{E}{\sim} N$

so that it is a congruence relation.

Let λE stand for $\lambda\beta\eta+\{M=N,$ for M,N in $E\}$.

By $\bar{E}$ we denote the *i-closure* of E, i.e. the set of all P such that $\lambda E \vdash M=P$, with M in E, that is the closure by identification of all elements of E.

DEFINITION 4 - E is *i-closed* iff $E = \bar{E}$.

For instance the set of unsolvable closed terms is i-closed; while $E=\{I,\Omega_2\}$ is not, because at least $\Omega_2\Omega_2$, $\Omega_2 I$ are in $\bar{E}$. It holds that if M and N are in $\bar{E}$ then $M \overset{E}{\sim} N$.

LEMMA 1 - $M \overset{E}{\sim} N$ iff $\lambda E \vdash M = N$.

Proof - The *if* part is trivial. The *only if* part, i.e. if $M \overset{E}{\not\sim} N$ then $\lambda E \not\vdash M = N$, holds because it is possible to build up a term model of $\lambda\beta\eta$ from the quotient set Λ_0/E (i.e. the set of equivalence classes) where all elements of E are equal and M does not become equal to N. Q.E.D.

Let's name i-property for a set E the following one:

(i) FM is in E iff FN is in E, for some F and every M,N in E.

THEOREM 1 - E is i-closed iff (i) holds for E.

Proof - If E is i-closed then obviously (i) holds.

If (i) holds and E is not i-closed then for some P in E and M in $\bar{E}$ but not in E, $M \overset{E}{\sim} P$ by lemma 1.

Hence there exists a finite sequence of terms M_i, with $M_0 \equiv M$ and $M_n \equiv P$, such that

$$M \overset{E}{\text{——}} M_1 \overset{E}{\text{——}} \dots\dots \overset{E}{\text{——}} M_{n-1} \overset{E}{\text{——}} P$$

and so also some M_i in E and M_{i+1} in $\bar{E}$ but not in E such that $M_i \overset{E}{\text{——}} M_{i+1}$, against the hypothesis. Q.E.D.

Remark - For instance the set of closed terms of order 0 (i.e. of terms of Λ_0 which never reduce to a term which can be written as $\lambda x.P$) is i-closed because it has the i-property: in fact if M and FM are of order 0 then $FMx_1...x_h$ is such that by reducing from it, $x_1,...,x_h$ are never disturbed by reduction, hence $FNx_1...x_h$ too has the same characteristics, in case N is of order 0, taking into account that avery recuction from $FMx_1...x_h$ (except those reductions, if any, performed from M) can be "simulated" by reducing from $FNx_1...,x_h$. (End of the remark).

We say that λE is inconsistent iff $\overline{E} = \Lambda_0$, so that all elements of E are consistently equable iff $\overline{E} \neq \Lambda_0$.

Since $\lambda\{S,K\} \vdash$ then for $\lambda E \vdash$ it is sufficient that $\lambda E \vdash S = K$. By lemma 1, $\lambda E \vdash$ iff $S \overset{E}{\sim} K$.

By this criterion the well known identifiability of all unsolvable terms, that is $\lambda\mathcal{U} \nvdash$, can be easely obtained. In fact, if $S \overset{\mathcal{U}}{\sim} K$ then there exists M_1 such that $M_1 \neq S$ and $S \overset{\mathcal{U}}{—} M_1$; but $FU = S$ and $FV = M_1$, with U,V in $\mathcal{U}$, implies, by a known result, that $Fx = S$ and so $S = M_1$.

DEFINITION 5 - M is *easy* iff $\lambda\{M,P\} \nvdash$, with arbitrary P. Hence M is easy if it is consistently equable to every P. Consider now the following property:

(e) If $F_1M = F_2M$ for some F_1, F_2, then there exists a $C[\ ,\ ,\ ]$ such that $F_1x \twoheadrightarrow C[x,M,x]$ and $F_2x \twoheadrightarrow C[M,x,x]$.

THEOREM 2 - If (e) holds for M then M is easy.

Proof - In fact, if the e-property holds for M then $S \overset{\{M,N\}}{\sim} K$ cannot be obtained for every N in Λ_0, analogously to what has been seen in [6] with $M \equiv \Omega_2$.

Let's define:

$P \overset{M,N}{\longleftarrow} Q$ (or $Q \overset{M,N}{\longrightarrow} P$) iff $P = FM$ and $Q = FN$ for some F in Λ_0.

It can be easily seen that:

I - $P \overset{\{M,N\}}{—} Q$ iff $P \overset{M,N}{\longleftarrow} Q$ or $P \overset{M,N}{\longrightarrow} Q$

II - $P \overset{M,N}{\longleftarrow} P$ and $P \overset{M,N}{\longrightarrow} P$

III - If $P \overset{M,N}{\longleftarrow} Q$, with $P \equiv S$ or $P \equiv K$, then $P = Q$ (It is sufficient to take $F_2 \equiv KP$)

IV - If $Q \xrightarrow{M,N} Q' \xleftarrow{M,N} P$ then there exists Q'' such that $P \xleftarrow{M,N} Q'' \xrightarrow{M,N} Q$. (From $Q = F_1N$, $Q' = F_1M = F_2M$, $P = F_2N$ it turns out $Q'' \equiv C[N,N,N]$, by (e)). Moreover let's call *alternated sequence* every sequence

$$P \xleftarrow{M,N} Q_1 \xrightarrow{M,N} Q_2 \xleftarrow{M,N} Q_3 . \; Q_{n-2} \xleftarrow{M,N} Q_{n-1} \xrightarrow{M,N} Q$$

where n is odd, the first arrow is left directed and consecutive arrows have alternated directions.

By I and III it follows:

V - If $S \overset{\{M,N\}}{\smile} K$ then there exists an alternated sequence between S and K.

Now it is easy to show that $S \overset{\{M,N\}}{\smile} K$ does not hold, since, by III and IV, an alternated sequence with a minimum n cannot exist. Q.E.D.

DEFINITION 6 - A set E of terms is *easy* iff every element of E is easy.

Now we are able to decide whether subsets of closed r. terms and of closed h.r. terms are *i*-closed or easy sets in the sense just specified.

3. i-closure

We denote by $\mathcal{R}$ and $\mathcal{HR}$ the subsets of Λ_0 respectively of r. and h.r. terms. Not every element of $\mathcal{HR}$ is of order 0, and this is the case also for every element of $\mathcal{R}$: for instance $\lambda x.\Omega_2\Omega_3$ and $\lambda x.\Omega_2$ are not of order 0. Let $\mathcal{R}_0 \subsetneqq \mathcal{R}$ stand for the set of o-r. terms, i.e. of r. terms which are also of order 0, and $\mathcal{HR}_0$ for the set of 0-h.r. terms.

It is well known that the set of unsolvable terms is i-closed. Moreover, the set of terms of order 0 is i-closed too, as we noticed by the remark.

In this paragraph we will see what is the case for $\mathcal{R}_0$, $\mathcal{R}$, $\mathcal{HR}_0$ and $\mathcal{HR}$.

Let $T \not\equiv U$ at most because some T', proper subterm of T, replaces all occurrences of U', proper subterm of U and of order 0. We shall say that a reduction $D: U \twoheadrightarrow V$ is *simulated up to* U' when we reduce from T by performing the same contractions, and in the same order, of D but by writing $T' \longrightarrow T'$ in correspondence to every contraction performed from U' (or from its reducta) in D. We shall also sometimes say that a reduction simulates another one.

LEMMA 2 - If T is 0-r. and $T \twoheadrightarrow C[T^{(n)}]$, then, by replacing $T^{(n)}$ by x, either $C[x] \twoheadrightarrow x$ or $C[x] \twoheadrightarrow T$.

Proof - Since T is r., there exists D: $C[T^{(n)}] \twoheadrightarrow T$, where the following two cases only are possible:

Case a. Every $T^{(k)}$, $k \geqslant n$, is erased inside D.

By simulating D from $C[x]$, up to the occurrences of $T^{(n)}$ replaced by x, a D': $C[x] \twoheadrightarrow T$ is obtained, since every x is erased too inside D'.

Case b. Some $T^{(k)}$, $k \geqslant n$, are not erased inside D.

T is of order 0, hence it is the descendant in D of a unique occurrence of $T^{(k)}$. Then, by reducing from $C[x]$ as in *case a*, the considered occurrence of $T^{(k)}$ is replaced by x and thus $C[x] \twoheadrightarrow x$. Q.E.D.

THEOREM 3 - If T, T_1 and T_2 are 0-r. and $FT_1 = T$, then FT_2 reduces to a 0-r. term.

Proof - By the remark, FT_2 is of order 0; now we have to show that it is r.

If $FT_1 = T$, with T 0-r. then $FT_1 \twoheadrightarrow T$, hence the set A_1 of r. reducta of FT_1 is not empty. Let A_2 be the set of reducta of FT_2 obtained by simulating all reductions from FT_1 to A_1.

Let Q_1 in A_1 be obtained by a reduction D_{11} and let Q_2 in A_2 be obtained by the reduction D_{21}, from FT_2, which simulates D_{11} up to $T_1^{(n)}$, $n \geqslant 0$ Let m be the number of $T_2^{(n)}$, $n \geqslant 0$, occurring in Q_2 which are not proper subterms of a reductum of T_1 or T_2.

Now, if D_{22}: $Q_2 \twoheadrightarrow P_2$ then by simulating D_{22} a D_{12}: $Q_1 \twoheadrightarrow P_1$, for some P_1, is obtained.

By hypothesis there exists D_{13}: $P_1 \twoheadrightarrow Q_1$, hence, by simulating D_{13}, there exists D_{23}: $P_2 \twoheadrightarrow Q_2'$. By lemma 2, $Q_2' \neq Q_2$ at most because of any occurrences of T_1 which have been created in D_{22} or D_{23} (and which of course have been created in D_{12} or D_{13} also) and which replace some occorrences of T_2.

If we suppose that Q_2 is one of the elements of A_2 with the minimum m, then $Q_2' = Q_2$, otherwise the minimality hypothesis for m fails Q.E.D.

COROLLARY 1 - $\mathcal{R}_0$ is i-closed.

Proof - By theorem 3, the i-property holds for $\mathcal{R}_0$, hence $\mathcal{R}_0 = \overline{\mathcal{R}}_0$. Q.E.D.

COROLLARY 2 - $\mathcal{HR}_0$ is not i-closed.

Proof - The i-property is not satisfied for $F \equiv \lambda x(\lambda yz.yyx)(\lambda yz.yyx)x$, $M \equiv \Omega_2\Omega_3$, and $N \equiv \Omega_2$. Q.E.D.

It is easy to see that $\Omega_2 \overset{\mathcal{HR}_0}{\sim} \lambda x.\Omega_2$, so that $\overline{\mathcal{HR}_0}$ is not the set of all unsolvable terms.

COROLLARY 3 - $\mathcal{R}$ is not i-closed.

Proof - Otherwise $\lambda x.\Omega_2 = \Omega_2$, i.e. $(\lambda x.\Omega_2)U = \Omega_2 U$, with U not r., i.e. $\Omega_2 = \Omega_2 U$, with $\Omega_2 U$ h.r. but not r. Hence (i) does not hold for instance with $F \equiv C_+\Omega_3$, where $C_+ \equiv \lambda xy.yx$, $M \equiv \Omega_2$, $N \equiv \Omega_2\Omega_3$. (Analogously for $\mathcal{R} - \mathcal{R}_0$ with $F \equiv C_+ I$, $M \equiv \lambda xy.\Omega_2$ and $N \equiv \lambda x.\Omega_2$). Q.E.D.

COROLLARY 4 - $\mathcal{HR}$ is not i-closed.

Proof - The same as for corollary 2. Q.E.D.

4. Ease

The set of terms of order 0 is not easy, as it is shown in [6] by $I = \Omega_3$.

THEOREM 4 - If $F_1 T = F_2 T$ with T 0-r. and F_1, F_2 arbitrary then there exists G such that $F_1 \equiv \lambda x.GxTx$ and $F_2 \equiv \lambda x.GTxx$.

Proof - By the Church-Rosser property, $F_1 T \twoheadrightarrow Q \twoheadleftarrow F_2 T$, with $Q \equiv C[T^{(n_1)}, T^{(n_2)}, \ldots, T^{(n_k)}]$, so that there exists $D_1 : F_1 T \twoheadrightarrow Q$ and $D_2 : F_2 T \twoheadrightarrow Q$. Inside every stage of D_1 and D_2 let's draw a circle around any subterm which reduces to T, with the proviso that any circle which should occur inside another one in a given stage, is erased in that stage.

Consider now $F_1 x$ and $F_2 x$ and draw a circle around any subterm which reduces either to T or to x. Let's reduce from $F_1 x$ and $F_2 x$ by simulating D_1 and D_2 respectively up to all subterms which are inside circles and at every stage draw a circle around any term which reduces either to T or to x. So doing D_1' and D_2' are obtained having also the same number of stages as D_1 and D_2 respectively.

Moreover let's say that two stages are *similar* when their circle-contexts, i.e. the contexts outside the circles, are identical.

Now we show, by induction on n, that stage n of D_1 and *stage n of D_1' are similar. The same holds for stage n of D_2* and stage n of D_2'.

First step - F_1T and F_1x are similar since they have the same number of circles and in the same position.

Inductive step - We show that if stages n of D_1 and D_1' are similar then so are stages $n+1$.

Consider what can happen by contracting from stage n to stage $n+1$ of D_1:

Case a. Some circles are erased or any new circle (i.e. which has been created by contracting) is a trace of some circle at stage n.

The same holds in D_1' and thus stages $n+1$ of D_1 and D_1' remain similar.

Case b. No fresh circle (i.e. which is not a trace of a circle already occurring in stage n of D_1) is built up.

Case b_1. The considered contraction concerns a term inside a circle. Then in D_1' the identity contraction is performed and stages $n+1$ of D_1 and D_1' are obviously similar.

Case b_2. The considered contraction concerns the circle-contexts.

The same contraction is performed in D_1' and thus stages $n+1$ of D_1 and D_1' are similar.

Case c. Some fresh circles are built up.

In such a case some subterms which reduce to T have been created. By lemma 2, the same contraction creates in D_1' correspondent subterms which reduce either to T or to x. Hence the same number of circles and in the same position occur in stages $n+1$ of D_1 and of D_1', so that, by the inductive hypothesis, stages $n+1$ of D_1 and of D_1' are similar.

All cases hold analogously for D_2 and D_2'.

Hence, by reducing from F_1x and F_2x, D_1': $F_1x \twoheadrightarrow Q_1'$ and D_2': $F_2x \twoheadrightarrow Q_2'$ are obtained where Q_1' is similar to Q and Q_2' is similar to Q, so that Q_1' and Q_2' are similar.

Moreover, subterms which occur inside circles in Q_1' and in Q_2' are reducible either to T or to x, so that, by further reducing, we can replace them by T or by x in an opportune way.

If we denote by Q_1 and Q_2 the terms so obtained from Q_1' and Q_2' respectively, we can write $F_1x \twoheadrightarrow Q_1$ and $F_2x \twoheadrightarrow Q_2$ where, for what we said, $Q_1 \equiv C[T,x,T,x]$ and $Q_2 \equiv C[T,T,x,x]$, Now, if $C[T,x_1,x_2,x_3]$ is such that: x_1 replaces all occurrences of circles which occur in the same position and which contain x in Q_1 and T in Q_2; x_2 replaces all occurrences of circles in the same position and which contain T in Q_1 and x in Q_2; finally x_3 replaces all occurrences of circles in the same position which in Q_1 and in Q_2 both contain x, then let $G \equiv \lambda x_1x_2x_3.C[T,x_1,x_2,x_3]$ and thus $F_1x = GxTx$ and $F_2x = GTxx$, that is $F_1 \equiv \lambda x.GxTx$ and $F_2 \equiv \lambda x.GTxx$. Q.E.D.

COROLLARY 5 - $\mathcal{R}_0$ is easy.

Anyway $\mathcal{R}_0$ loses the ease property in case all its elements are equated, since $I = \Omega_2 = \Omega_2K = IK = K$.

COROLLARY 6 - $\mathcal{HR}_0$ is easy.

Proof - Let $RU_1 \ldots U_h$ be an element of $\mathcal{HR}_0$.

Since R is easy, let $R \equiv K^nM \equiv \underbrace{K(K(\ldots(KM)}_{n}\ldots))$, with M arbitrary closed term. Then $RU_1 \ldots U_h = M$. Q.E.D.

COROLLARY 7 - $\mathcal{R}$ is not easy.

Proof - Otherwise from $\lambda x.\Omega_2 = I$, $\lambda x.Q_2P = IP$ for arbitrary P is obtained, i.e. $\Omega_2 = P$, from which $S = K$ follows. Q.E.D.

Hence $\mathcal{HR}$ is not easy.

5. Ease and i-closure for CL

Since the equivalence between h.n.f. and solvability fails in CL, $\mathcal{R} - \mathcal{R}_0$ is a subset of the set of terms in h.n.f., if to be in h.n.f. means, for a term, to be without head redex. Analogously for $\mathcal{HR} - \mathcal{HR}_0$.

Moreover for WCL with S,K,I, the set of terms in n.f. (and hence those in h.n.f., r., h.r., ecc.) do not correspond, by every transformation function, to the sets having the same definition in $\lambda\beta\eta$. Nevertheless the definitions of r. and h.r. term are the same, so that by reading all properties mentioned in this paper in CL or WCL, the theorems here proved do hold, for the appropriate set of terms, in CL and also in WCL, since their proofs have been carried out without using neither rule (ξ) nor the η-rule nor extensionality rules.

Conclusions

We simply sum up what is the situation about ease and i-closure for some sets of unsolvable terms which are subsets of Λ_0, C_0 and WC_0 respectively.

Λ_0 or C_0 or WC_0	i-closed	easy
$\mathcal{U}$ (set of unsolvable terms)	YES	NO
$\mathcal{U}_0$ (set of unsolvable terms of order 0)	YES	NO
$\mathcal{HR}_0$ (set of head recurrent terms of order 0)	NO	YES
$\mathcal{R}$ (set of recurrent terms)	NO	NO
$\mathcal{R}_0$ (set of recurrent terms of order 0)	YES	YES

Open problems

I - Define an i-closed and easy set including $\mathcal{R}_0$.

II - Is the set of all easy elements an i-closed one?

III - Characterize the set of easy terms.

IV - Characterize the i-closure of $\mathcal{HR}_0$.

V - Is $\overline{\mathcal{HR}_0}$ easy?

VI - Does every easy term satisfy (e)?

VII - Define a set as in I and recusively enumerable.

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