~~PRIVATE COMMUNICATION~~ 711115 C.B.

~~SOME REMARKS ON THE SEMIGROUP OF~~

THE COMBINATOR SEMIGROUP. HAS 2 GENERATORS

Church proved [1] that 4 elements generate the whole set $\mathcal{C}$ of Combinators [closed formulae of the λ-K-calculus] as semigroup, with unsolvable word problem, taking as semigroup operation B (or CB). In [2] I observed that the generators can be reduced to 3. Here it is proved that there exist two generators for $\mathcal{C}$. This result (in some sens the best one)(1) could mean something if we are interested in the semigroup structure of $\mathcal{C}$. ~~There is some evidence that the semigroup operation CB [indicated below as infix by $\circ$] is better than B for application in the theory of computation and also more appropriate for an axiomatic approach to a $\mathcal{C}$-theory as semigroup with one operator.~~

(1) It cannot exist only one generator Q: $Q^h = I$ (h integer) implies together with $I \circ x = x$ that ~~non convertible combinators~~ at most a finite number (h) of mutually non convertible combinator could exist.

THEOREM.

$H \equiv (C_* S) \circ (C_* (K \circ (C_* K))) = \lambda x (x (BK((C_* K)) S))$ and $C_* = \lambda x \lambda y (yx)$ generate $\mathscr{C}$, under composition $\circ$ as (semigroup) operation. In other words

$$\mathscr{C} = (H, C_* ; \circ)$$

<u>Proof</u> Church's convertibility relations

1) $C_*(xy) = (C_* y) \circ (C_* x) \circ B$

2) $C_* Z \circ C_* = Z$

are sufficient to prove, by structural induction, that

$$\mathscr{C} = (C_* K, C_* S, C_*, B ; \circ)$$

Using two more relations, ~~easily~~ easily checked,

3) $B = S \circ K$

4) $C_* t \circ K = I$ (~~I is the identity~~)

we can prove using 2) twice

$$B = (C_* S) \circ C_* \circ (C_* K) \circ C_*$$

hence $\mathscr{C} = (C_* K, C_* S, C_* ; \circ)$

Now

(5) $H \circ K = C_* S \circ (C_*(\cdots)) \circ K = C_* S$ by (4) and

(6) $H \circ C_* = C_* S \circ (C_*(K \circ (C_* K)) \circ C_* =$ by (2)

$= C_* S \circ K \circ C_* K =$ by (4)

$= C_* K$

Using again (6) and (2), (5) can by written as

(7) $C_* S = H \circ (C_* K) \circ C_* = H^2 \circ C_*^2$.

From (6) and (7) it follows that

$$\mathscr{C} = (H, C_* ; \circ) \qquad \text{q. e. d.}$$

Corrado Böhm

[1] Church, A.

[2] Böhm, C. "Alcune proprietà delle forme β-η-normali nel λ-K-calcolo", Pubbl. Ist. Naz. Appl. Calcolo n. 696 - CNR, ROMA (1968)

THE COMBINATOR SEMIGROUP HAS 2 GENERATORS

Church proved [1] that 4 elements generate the whole set $\mathcal{C}$ of combinators (closed formulas of the λ-K-calculus) as semigroup, with unsolvable word problem, taking as semigroup operation B (or CB) . In [2] I observed that the generators can be reduced to 3. Here it is proved that there exist two generators for $\mathcal{C}$. This result (in some sens the best one)[1] ~~something~~ could mean somethig if we are interested in the semigroup structure of $\mathcal{C}$.

THEOREM. $H=(C_*S)\circ(C_*(K\circ(C_*K)))=\lambda x(x(BK(C_*K))S)$ and $C_*=\lambda x\lambda y(yx)$ generate $\mathcal{C}$, under composition $\circ$ as semigroup operation. In other words

$$\mathcal{C}=(H,C_*;\circ)$$

<u>Proof</u> . Church's convertibility relations

1) $C_*(xy)=(C_*y)\circ(C_*x)\circ B$

2) $C_*x\circ C_*=x$

are sufficient to prove, by structural induction, that

$$\mathcal{C}=(C_*K,C_*S,C_*,B;\circ) .$$

Using two more relations, easily checked,

3) $B=S\circ K$

4) $C_*I\circ K=I$

we can prove using 2) twice

$$B=(C_*S)\circ C_* \circ(C_*K)\circ C_*$$

hence

$$\mathcal{C}=(C_*K,C_*S,C_*;\circ) .$$

Now

5) $H\circ K=C_*S\circ(C_*(\ldots))\circ K=C_*S$ by 4) and

6) $H\circ C_*=C_*S\circ(C_*(K\circ(C_*K))\circ C_*=$ by 2)

$=C_*S\circ K\circ C_*K=$ by 4)

$=C_*K$

Using again 6) and 2), 5) can be rewritten as

7) $C_*S=H\circ(C_*K)\circ C_*=H^2\circ C_*^2$.

From 6) and 7) it follow that

$$\mathcal{C}=(H,C_*;\circ)$$

Q.E.D.

§1) It cannot exist only one generator Q: Q^h=I (h integer) implies together with I.x=x that at most a finite number (h) of mutually not convertible combinators could exist.

BIBLIOGRAFIA

1 Church, A. "Combinatory logic as a semigroup" (abstract) Bull. Amer. Math. Soc. 43: 333 (1937)

2 Böhm, C. "Alcune proprietà delle forme β-η-normali nel λ-K-calcolo" Pubbl. Ist. Naz. Appl. Calcolo N; 696 C.N.R. ROMA (1968)