

~~PRIVATE COMMUNICATION~~

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~~SOME REMARKS ON THE SEMIGROUP OF~~THE COMBINATOR SEMIGROUP. ~~THIS~~ HAS 2 GENERATORS  
4 elementsChurch proved [1] that ~~CB~~ ~~CCB~~ ~~CCC~~ ~~CCCC~~ generatethe whole set  $\mathcal{C}$  of combinator<sup>s</sup> (closed under 26663 formulae of the 2-K-calculus) as semigroups, with

unsolvable word problem, taking as semigroup operation

I observed

B (or CB). In [2] ~~it is proved~~ that the generators

it is proved

can be reduced to 3. Here ~~it is proved~~ that there existtwo generators for  $\mathcal{C}$ . (in some sense the best one) This result could mean

something if we are interested in the semigroup structure

of  $\mathcal{C}$ .There is some evidence that the ~~semigroup~~  
~~operator~~  
~~Ole~~CB [indicated below as infix by  $\cdot$ ] is ~~operator~~  
~~better than B~~  
better for application in the theory of computation.and also more appropriate for an axiomatic approach  
to a  $\mathcal{C}$ -theory as semigroup with one operator.

~~THE FIRST ALLEGATION~~<sup>(1)</sup> It cannot exist only one generator? ~~while~~  
 following ~~it follows~~  $Q^k = I$  <sup>( $k$  integer)</sup> implies together with  $I \circ x = x$   
 that ~~only~~ ~~one~~ ~~non~~ ~~convertible~~ ~~combinator~~  
~~exists~~ at most a finite number ( $h$ ) of mutually non  
 convertible combinator could exist.

THEOREM.

$H \equiv (C_x S) \circ (C_x (K \circ (C_x K)))$  and  $C_x$  generate  $C$ ,  
under composition  $\circ$  as <sup>semigroup</sup> operation. In other words  
 $C = (H, C_x; \circ)$

Proof

Church's convertibility relations

$$1) \quad C_x(xy) = (C_x y) \circ (C_x x) \circ B$$

$$2) \quad C_x z \circ C_x = z$$

are insufficient to prove, by structural induction, that

$$C = (C_x K, C_x S, C_x, B; \circ)$$

Using two more relations, ~~are~~ easily checked,

$$3) \quad B = S \circ K$$

$$4) \quad C_x t \circ K = I \quad (\text{Is this so?})$$

we can prove using 2) twice

$$B = (C_x S) \circ C_x \circ (C_x K) \circ C_x$$

hence

$$C = (C_x K, C_x S, C_x; \circ)$$

Now

$$(5) \quad H \circ K = C_x S \circ (C_x(\dots)) \circ K = C_x S \quad \text{by (4) and}$$

$$\begin{aligned} (6) \quad H \circ C_x &= C_x S \circ (C_x(K \circ (C_x K)) \circ C_x) = && \text{by (2)} \\ &= C_x S \circ K \circ C_x K &=& \text{by (4)} \\ &= C_x K \end{aligned}$$

Using again (6) and (2), (5) can be written as

$$(7) \quad C_x S = H \circ (C_x K) \circ C_x = H^2 \circ C_x^2.$$

From (6) and (7) it follows that

$$C = (H, C_x; \circ) \quad \text{q. e. d.}$$

$$H \circ K = C^* \circ (C^* \circ C_x; \circ) \circ K = C^* C_x \circ C_x; \circ$$

Corrado Böhm

$$S = (C^* K) C^* C_x; \circ$$

$$B = C^* C_x; \circ C_x; \circ$$

[1] Church, A.

[2] Böhm, C. "Alcune proprietà delle forme  $\beta\eta$ -normali  
nel  $\lambda$ -K-calcolo", Pubbl. Ist. Naz. Appl. Calcolo  
n. 696 - GNR, ROMA (1968)

$$S = (C^* K) C^* C_x; \circ C_x; \circ$$

$$\rightarrow C^* B \circ C^* = S$$

$$\text{cioè } C^* (B \circ S) = (C^* B) \circ (C^* S) \circ B$$

Proof:  $C^*$  è un  $\lambda$ -calcolo composto da operazioni

$$S = (H, C_x; \circ)$$

composte da applicazione e formule, che sono operazioni

$H \in \text{Cer}(\lambda\text{-calcolo})$  e  $C_x$  formule di  $\lambda$

LHEOBEN

$(C^* B) \circ (C^* S) \circ B$

THE COMBINATOR SEMIGROUP HAS 2 GENERATORS

Church proved [1] that 4 elements generate the whole set  $\mathcal{C}$  of combinators (closed formulas of the  $\lambda$ -calculus) as semigroup, with unsolvable word problem, taking as semigroup operation  $\circ$  (or  $\circ\circ$ ). In [2] I showed that the generators can be reduced to 3. Here it is proved that there exist two generators for  $\mathcal{C}$ . This result (in some sense the best one)<sup>(1)</sup> something could even something if we are interested in the semigroup structure of  $\mathcal{C}$ .

LEMMA.  $B = (0_x \beta)_x (0_x (\lambda_x (0_x x))) = \lambda x (x (\lambda x (0_x x)))$  and  $C_x = \lambda x y (yx)$  generate  $\mathcal{C}$ , under composition  $\circ$  as semigroup operation. In other words

$$\mathcal{C} = (B, C_x)^\circ$$

Proof. Church's convertibility relations

- 1)  $0_x (xy) = (0_x y)_x (0_x x) \circ B$
- 2)  $0_x x \circ 0_x = x$

are sufficient to prove, by structural induction, that

$$\mathcal{C} = (0_x \beta, 0_x S, 0_x K, 0_x I)^\circ$$

Using ten more relations, easily checked,

- 3)  $B \circ B = K$
- 4)  $C_x \circ C_x = I$

we can prove using 2) twice

$$B = (0_x \beta)_x 0_x \circ (0_x \beta) \circ 0_x$$

hence

$$\mathcal{C} = (0_x \beta, 0_x S, 0_x K, 0_x I)^\circ$$

Now

- 5)  $B \circ B = 0_x (0_x (\dots)) \circ 0_x = 0_x \beta$  by 4) and
- 6)  $B \circ 0_x = 0_x \circ B = (0_x (\lambda_x (0_x x))) \circ 0_x = -0_x \circ 0_x \beta = -0_x \circ 0_x$  by 2)
- 7)  $= 0_x \circ 0_x$  by 4)
- 8)  $= 0_x I$

Using again 6) and 7), 5) can be rewritten as

$$7) 0_x \circ B = (0_x \beta)_x 0_x \circ I^2 = 0_x^2$$

From 6) and 7) it follow that

$$\mathcal{C} = (B, C_x)^\circ$$

Q.E.D.

§1) It cannot exist only one generator  $Q: Q^h = I$  ( $h$  integer) implies together with  $I \circ x = x$  that at most a finite number ( $h$ ) of mutually not convertible combinators could exist.

#### BIBLIOGRAFIA

- 1 Church, A. "Combinatory logic as a semigroup" (abstract) Bull. Amer. Math. Soc. 43:333 (1937)
- 2 Böhm, C. "Alcune proprietà delle forme  $\beta\text{-}\eta$ -normali nel  $\lambda\text{-}\kappa$ -calcolo" Pubbl. Ist. Naz. Appl. Calcolo N. 696 C.N.R. ROMA (1968)