Henk Barendregt
April 1970
13 pages

A universal generator for the λ-calculus

By Henk Barendregt

§ 1 Introduction

§ 0 By λ-calculus we mean the λ-K calculus with β-conversion. See e.g. Curry Feys [2], p 91.
The reduction relation will be denoted by $\geq$, the metamathematical equality by $\equiv$, and convertibility by $=$.

(2)

Def. The *family* of a formula M of the λ-calculus is the set of all formulas N, such that N is a subformula of a reduction of M.

Def. A formula is called a universal generator if

(bijlage)

[illegible]

As is well known, there are infinite decending chains with respect to the relation $\geq$. For example define

$\Omega = (\lambda x.xx)(\lambda x.xx)$ and
$\Omega' = (\lambda x.xx(xx))(\lambda x.xx(xx))$ then

naïeve regels) $\Omega \geq \Omega \geq \Omega \ldots$ and
linearie regels) $\Omega' \geq \Omega'\Omega' \geq \Omega'\Omega'(\Omega'\Omega')$ etc.

In the case of Ω' the reducts are even growing indefinitely.

In this paper we will construct a term M, such that its reducts will (~~get~~ have? ~~acquire?~~ ... arbitrary ~~length and shape~~ complexity

'get' is slangy, but I think it is allowable. otherwise alts: 'have', 'acquire', 'receive', 'attain' its family consists of all ~~formulas~~ terms of the λ-calculus.

In § 2 we will construct a universal generator. In order to do this we need to restate some results of Kleene's (in a rather ~~trivial~~ obvious way) This is done in § 1.

§ 1. The enumeration theorem.

In the λ-calculus we can define numerals:

$$\underline{0} \equiv \lambda ab.\, b$$

$$\underline{1} \equiv \lambda ab.\, ab$$

$$\underline{2} \equiv \lambda ab\ a(ab)$$

$$\underline{3} \equiv \lambda ab.\, a(a(ab))$$

etc.

In general

$$\underline{n} \equiv \lambda ab\ \underbrace{a(a \cdots a(ab) \cdots)}_{n \text{ times}} \equiv \lambda ab.\, a^n b$$

1.2 Definition A numbertheoretic function f (of k arguments) is called λ-definable if there exists a ~~formula~~ term F such that

$$f(n_1, \ldots, n_k) \equiv m \iff F\underline{n_1} \ldots \underline{n_k} = \underline{m}$$

A classical result of Church, Kleene and ~~Rosser~~ is:

1.3 Theorem A function f is λ-definable iff f is ~~partial~~ recursive.

See for a proof Kleene [1936].

1.4. Corollary For every partial recursive function f there exists a ~~formula~~ term F such that

$$f(n_1, \ldots, n_k) \equiv m \iff F\underline{n_1} \ldots \underline{n_k} \geqslant \underline{m}$$

Proof: f is λ-definable, hence there exists a ~~formula~~ term F such that

$$f(n_1, \dots, n_k) \equiv m \Leftrightarrow F\,\underline{n_1} \dots \underline{n_k} = \underline{m}$$

By the Church-Rosser theorem (see e.g. Curry-Feys [1958], pp. 109-117.)

$F\,\underline{n_1} \dots \underline{n_k} = \underline{m}$ iff $F\,\underline{n_1} \dots \underline{n_k} \geq M$ and $\underline{m} \geq M$ for some M.

But $\underline{m}$ is in normal form, hence $\underline{m} \equiv M$ (up to α-convertibility), which yields $F\,\underline{n_1} \dots \underline{n_k} \geq \underline{m}$. $\boxtimes$

Let τ be a recursive ordered pair-function for the integers with τ_1, τ_2 as first resp. second coordinate functions, such that $\tau(x,y) \geq 2$.

Let $x \dot{-} y \equiv \begin{cases} x-y & \text{if } x>y \\ 0 & \text{else} \end{cases}$

Let S^+, $\lambda xy\,[x \dot{-} y]$, T, T_1, T_2 be the defining ~~formulas~~ terms for resp the successor function, the function $x \dot{-} y$, τ, τ_1 and τ_2.

1.5. **Definition** $K \equiv (\lambda ab\,.\,a)$

(inverse regd)

$S \equiv (\lambda abc\,.\,ac(bc))$

1.6. **Definition** A combinator is an element of the least class $\mathbf{C}$ of ~~formulas~~ terms such that

1) $K \in \mathbf{C}$, $S \in \mathbf{C}$

2) If $M \in \mathbf{C}$ and $N \in \mathbf{C}$ then $MN \in \mathbf{C}$.

1.7 **Lemma** Cor For every formula (without free variables) M, there exists a combinator M_H such that $M_H \geq M$.

For a proof see e.g. Curry Feys [1958] Ch 6

1.8. Lemma ~~1.4~~ For all ~~formulas~~ terms A_0, A_1 there exists a ~~formula~~ terms L such that

$$L\underline{0} \geqslant A_0$$

$$L\underline{1} \geqslant A_1$$

Proof.

Define $L \equiv (\lambda a.\ a(KA_1)A_0)$. $\boxtimes$

1.9. Lemma ~~1.4~~ For all ~~formulas~~ terms $A_0, \dots, A_n$ there exists a ~~formula~~ term L such that

$$L\underline{i} \geqslant A_i \quad \text{for } 0 \leq i \leq n.$$

Proof: ~~By~~ Induction on n

If $n \equiv 0$ then define $L \equiv KA_0$

Suppose $n \equiv k+1$

According to the induction hypothesis there exists a ~~formula~~ term L_0 such that

$$L_0\underline{0} \geqslant A_1, \dots, L_0\underline{k} \geqslant A_{n+1}.$$

Let L_1 be such that $L_1\underline{0} \geqslant A_0$

Let G be such that

$$g\,\underline{0} \geq L_0$$

$$g\,\underline{1} \geq L_1 .$$

Define $L \equiv \lambda i\; g[\underline{1} \dot{-} i][i \dot{-} \underline{1}]$, then L has the required properties. ⊠

1.10 Theorem: There exists a formula E, such that for any combinator M there exists an integer n such that

$$E\,\underline{n} \geq M$$

Proof: Define $V \equiv \lambda u i\;\; u(S^{+}(\underline{2} \dot{-} i))(i \dot{-} \underline{1})u$

From 1.9. it follows that there exists a U such that:

$$U\underline{0} \geq \lambda x.x$$

$$U\underline{1} \geq \lambda xy.(Vy(T_1(x \dot{-} \underline{1})))(Vy(T_2(x \dot{-} \underline{1})))$$

$$U\underline{2} \geq \lambda xy.\, yxK$$

$$U\underline{3} \geq \lambda xy.\, yxS$$

Define $E \equiv \lambda i. VUi$

Then ~~E has the properties~~

$$E\underline{0} \geq VU\underline{0} \geq U\underline{3}\,\underline{0}U \geq U\underline{0}S \geq S$$

$$E\underline{1} \geq VU\underline{1} \geq U\underline{2}\,\underline{0}U \geq U\underline{0}K \geq K$$

(*) $$\begin{cases} E\underline{n+2} \geq U\underline{1}\,\underline{n+1}U \geq (VU(T_1\underline{n}))(VU(T_2\underline{n})) \\ \equiv E(T_1\underline{n})(E(T_2\underline{n})) \end{cases}$$

Now we prove with induction to the ~~length~~ complexity of a combinator M that there exists a ~~natural~~ number ~~m~~ $\underline{n}$ such that $E\underline{n} \geq M$

If $M \equiv K$ or $M \equiv S$ then $\underbrace{\text{}}_{}$(we can take) $\underline{n} \equiv \underline{0}$ or $\underline{n} \equiv \underline{1}$

Suppose $M \equiv M_1 M_2$ and

$$E\underline{n_i} \geq M_i \qquad i \equiv 1,2.$$

Then $\tau(n_1, n_2) \geq 2$ and it follows from (*) and 1.4. that

~~Then it follows from (*) and cor 1.2. that~~

$$\cancel{\ldots}\ E\ \underline{I(n_1, n_2)} \geq$$

$$E(T_1(\underline{I(n_1, n_2)}))(E(T_2(\underline{I(n_1, n_2)}))) \geq$$

$$E\underline{n_1}\ E\underline{n_2} \geq M_1 M_2 \equiv M \;.\ \boxtimes$$

1.8, 1.9 and 1.10 are slight modifications of results of Kleene's as found in Church [1 1941], p46-48

§2 A universal generator.

21 Lemma There are ~~formulas~~ terms D, D_1, D_2 such that for all ~~formula~~ terms M_1, M_2

$$D_i(D M_1 M_2) \geq M_i \qquad i = 1, 2.$$

Proof: Define

$$D \equiv (\lambda abc.\ c\, ab)$$

$$D_1 \equiv \lambda a.\ a\,(\lambda bc.\, b)$$

$$D_2 \equiv \lambda a.\ a\,(\lambda bc\ c).\ \boxtimes$$

Notation: Dxy will be denoted by $[x, y]$.

22 Lemma There exists a formula FP such that for all M

$$FP\, M \geq M(FP\, M)$$

Proof: Define $FP \equiv \lambda x.[(\lambda y.\, x(yy))(\lambda y.\, x(yy))].\ \boxtimes$

2.3 Theorem ~~III~~ There exists a universal generator for the λ-calculus.

Proof: Define
$$A \equiv \lambda xy.\, x[E(S^+ y), S^+ y]$$
$$B \equiv \lambda xy\, Ax(D_2 y)$$
$$X_0 \equiv FP\, B$$
$$G \equiv X_0[E\underline{0}, \underline{0}].$$

Here E stands for the ~~formula~~ term of theorem 1.10.

Note that $X_0 \geq BX_0 \geq \lambda y.\, AX_0(D_2 y)$

Hence:

$$G \equiv X_0[E\underline{0}, \underline{0}] \geq AX_0(D_2[E\underline{0}, \underline{0}]) \geq AX_0\,\underline{0} \geq$$
$$X_0[E\underline{1}, \underline{1}] \geq AX_0(D_2[E\underline{1}, \underline{1}]) \geq AX_0\,\underline{1} \geq$$
$$X_0[E\underline{2}, \underline{2}] \geq \quad \text{etc.}$$

From theorem 1.10. it follows that G has every combinator in its family. By 1.7 it follows that G has every ~~formula~~ term without free variables in its family. Because any ~~formula~~ term M is a subformula of a ~~formula~~ term without free variables (e.g. $\lambda x_1 \ldots x_n . M$) G is a universal generator. ⊗

References

[1] Church, A [1941] The calculi of λ-conversion. Ann of Math Studies 6 Princeton N.J

[2] Curry, H.B and R. Feys [1958] Combinatory Logic, ~~North-Holland Co~~ Amsterdam.

[3] Kleene, S.C. [1936] λ-definability and recursiveness, Duke Math Journ., Vol 2 ~~[illegible]~~, pp. 340 – 353

Utrecht, April 1st, 1970.
~~Amsterdam 1-4-1970.~~

Modification of D. Scott (at a dinner in Amsterdam, ~~17-4-1970)~~ April 17th, 1970).

Define $B = \lambda b n . [En, b(S^+ n)]$

$b_0 = FP\,B$, where E, S^+ and $[\;]$ are the same as in 2.3

Then $b_0 \underline{n} \geq B b_0 \underline{n} \geq [E\underline{n}, b_0 \underline{n+1}]$

Hence $b_0 \underline{0} \geq [E\underline{0}, b_0 \underline{1}] \geq [E\underline{0}, [E\underline{1}, b_0 \underline{2}]] \geq \cdots$. ~~is~~

Therefore ($b_0 \underline{0}$ is) a universal generator.